Unit 20

Circular motion

Introduction

The theme of this unit is rotational motion. We will concentrate mainly on analysing the circular motion of a particle. This can be used to model a wide range of situations, such as a child on a swing, the pendulum of a clock and a chair-o-plane roundabout (see Figure 1) at the fairground.

Figure 1 Chair-o-plane roundabout

Many fairground rides depend for their thrills on the forces involved in circular motion. We will be able to use our model of the chair-o-plane, for example, to answer questions such as the following.

- Will a child swing out at a greater angle than a heavier adult?
- Will the people on the outside swing out at a greater angle than those on the inside?
- How does the angle at which people swing out depend on the speed of rotation of the roundabout?

These questions will be answered in Subsection 2.5.

Before we are able to answer questions such as these, we need to investigate the motion of a particle moving in a circle. In Section 1 we derive expressions for the velocity and acceleration of a particle moving in a circle. In Section 2 we apply Newton's second law to uniform circular motion, that is, motion in a circle with constant speed. In Section 3 this is extended to non-uniform circular motion. Section 4 starts to look *beyond* circular motion to the concepts that are needed in order to analyse more general rotational motion.

1 Describing circular motion

Consider an object following a circular path at constant speed. This might be a particle attached to an inextensible string swung round in a circle, or a vehicle following a circular path on a roundabout or part of a circle when cornering. In this section we analyse this motion. We do so first in Cartesian coordinates and subsequently in polar coordinates, after an intervening subsection revising the differentiation of vector functions.

1.1 Circular motion in Cartesian coordinates

In this subsection we take a first look at circular motion, using Cartesian coordinates. Some simple features of the motion will be derived, which can be compared with more general results derived later.

Recall that the point $(\cos\theta, \sin\theta)$ lies on a circle of radius 1 and centre $(0, 0)$. More generally, $(R\cos\theta, R\sin\theta)$ lies on a circle of radius R and centre $(0, 0)$. If t denotes time and α is a constant, then a particle with position

$$(R\cos(\alpha t), R\sin(\alpha t))$$

at time t will move on a circle of radius R with centre $(0, 0)$. The line joining the particle's position to $(0, 0)$ makes an angle $\theta = \alpha t$ with the positive x-axis, and this angle is changing with time at a constant rate α, that is, $\dot{\theta} = \alpha$. The quantity $\omega = |\alpha| = |\dot{\theta}|$ is the **angular speed** of the particle. (The definition $\omega = |\dot{\theta}|$ applies even when $\dot{\theta}$ is not constant.)

Angular speed has units of radians per second.

The motion considered in this subsection, where $\dot{\theta}$ is constant, is called **uniform circular motion**. If the motion is circular and $\dot{\theta}$ is not constant, then the motion is called **non-uniform circular motion**.

To study the mechanics of a particle moving in a circle at constant angular speed, we need to know its velocity and acceleration.

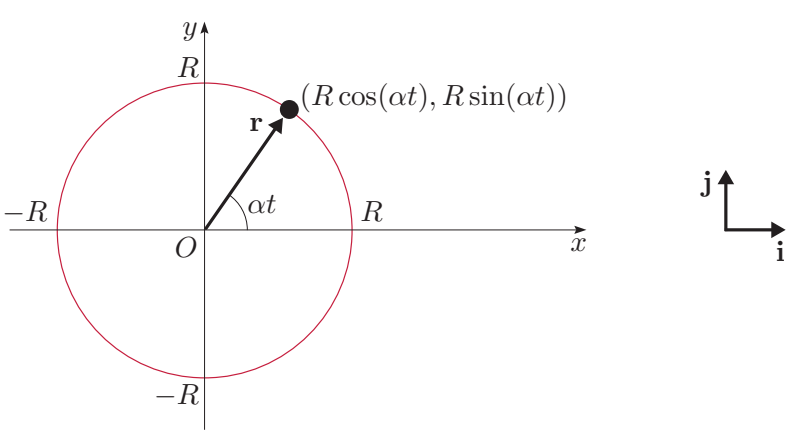


Figure 2 Position vector of a particle moving in a circle

Choosing Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ as shown in Figure 2, the position vector of the particle is given by

$$\mathbf{r} = R\cos(\alpha t)\mathbf{i} + R\sin(\alpha t)\mathbf{j}. \tag{1}$$

Before looking at the general case of a particle moving uniformly in a circle, try the following exercise.

We have

$$|\mathbf{r}| = \sqrt{R^2\cos^2(\alpha t) + R^2\sin^2(\alpha t)} = R,$$

which is the radius of the circle.

Exercise 1

A particle has position vector $\mathbf{r}$ at time t given by

$$\mathbf{r}(t) = 3\cos(4t)\mathbf{i} + 3\sin(4t)\mathbf{j}.$$

(a) Find the velocity vector $\dot{\mathbf{r}}(t)$ and the acceleration vector $\ddot{\mathbf{r}}(t)$ of the particle at time t.

(b) What is the magnitude of the acceleration vector $\ddot{\mathbf{r}}(t)$?

(c) Show that $\ddot{\mathbf{r}}(t)$ is of the form $-c\,\mathbf{r}(t)$, where c is a positive constant.

Now consider again the case of a particle with position vector given by equation (1), and differentiate to obtain the particle's velocity

$$\begin{aligned}\mathbf{v} = \dot{\mathbf{r}} &= -\alpha R\sin(\alpha t)\mathbf{i} + \alpha R\cos(\alpha t)\mathbf{j} \\ &= \alpha R(-\sin(\alpha t)\mathbf{i} + \cos(\alpha t)\mathbf{j}).\end{aligned} \tag{2}$$

Note that R is a constant (the radius of the circle).

What can we deduce about the velocity from this expression? First, $\mathbf{v}$ has magnitude

$$|\mathbf{v}| = \sqrt{\alpha^2R^2\sin^2(\alpha t) + \alpha^2R^2\cos^2(\alpha t)} = |\alpha|R = \omega R.$$

So the particle moves with speed ωR. Since both ω and R are independent of t, the speed of the particle is constant. (The *velocity* is not constant because its direction is changing.) Second, notice that

$$\begin{aligned}\mathbf{r}\cdot\mathbf{v} &= \big(R\cos(\alpha t)\mathbf{i} + R\sin(\alpha t)\mathbf{j}\big)\cdot\big(-\alpha R\sin(\alpha t)\mathbf{i} + \alpha R\cos(\alpha t)\mathbf{j}\big) \\ &= -\alpha R^2\cos(\alpha t)\sin(\alpha t) + \alpha R^2\cos(\alpha t)\sin(\alpha t) \\ &= 0.\end{aligned}$$

You saw in Unit 2 that if the dot product of two non-zero vectors is zero, then they are perpendicular. So the particle's velocity $\mathbf{v}$ is perpendicular to its position vector $\mathbf{r}$. This is illustrated in Figure 3, which shows that the velocity $\mathbf{v}$ is tangential to the circle that the particle is tracing.

Differentiating the velocity vector $\mathbf{v}$ (equation (2)) gives the acceleration

$$\begin{aligned}\mathbf{a} = \ddot{\mathbf{r}} &= \alpha R(-\alpha\cos(\alpha t)\mathbf{i} - \alpha\sin(\alpha t)\mathbf{j}) \\ &= -\alpha^2R(\cos(\alpha t)\mathbf{i} + \sin(\alpha t)\mathbf{j}) \\ &= -\alpha^2\mathbf{r} \\ &= -\omega^2\mathbf{r}.\end{aligned} \tag{3}$$

Since this vector is a negative multiple of $\mathbf{r}$, the acceleration is towards the centre of the circle (see Figure 3).

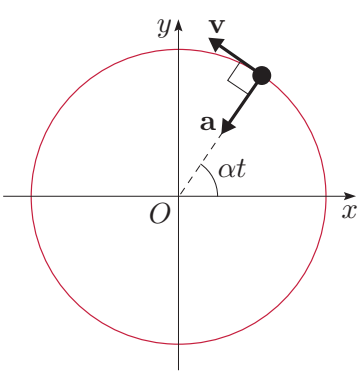


Figure 3 Velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of a particle moving in a circle at constant angular speed

The magnitude of the acceleration is $\omega^2 R$. Since $|\mathbf{v}| = \omega R$, the magnitude of the acceleration can also be written as $|\mathbf{v}|^2/R$. This form is useful if we know the speed of the particle, rather than its angular speed.

The main results derived above are summarised below.

If a particle follows a circle of radius R at a constant speed $|\mathbf{v}|$, then it has a constant angular speed $|\mathbf{v}|/R$, and these results apply.

Motion in a circle at constant angular speed

For a particle moving in a circle of radius R with constant angular speed ω:

- the speed (the magnitude of the velocity) is a constant, $|\mathbf{v}| = \omega R$
- the direction of the velocity is tangential to the circle
- the acceleration has constant magnitude, $|\mathbf{a}| = \omega^2 R = |\mathbf{v}|^2/R$
- the direction of the acceleration is towards the centre of the circle.

Since a particle moving in a circle in this way has a non-zero acceleration, it must be subject to some force, say $\mathbf{F}$. By Newton's second law, this force must act in the same direction as the acceleration $\mathbf{a}$, that is, towards the centre of the circle. Such a force might be provided by a string attaching the particle to the centre of the circle. Alternatively, if the particle is constrained to move in a circle by following a track, then the track must exert a sideways force on the particle (the force acting towards the centre of the circle). If the particle has mass m, then Newton's second law gives $\mathbf{F} = m\mathbf{a} = -m\omega^2\mathbf{r}$. The force is towards the centre of the circle and has magnitude $m\omega^2 R$, or $m|\mathbf{v}|^2/R$.

The results listed in the summary box above are sufficient to solve some problems involving circular motion, such as the following exercise.

Exercise 2

A miniature locomotive of mass 40 kg is following a circular track of diameter 200 m at constant speed. It completes one circuit in 50 s.

What is the magnitude of the sideways force exerted by the track on the locomotive?

In the form of motion discussed in this subsection, we have a particle subject to a force at right angles to its direction of motion. The effect of this force is not to change the magnitude of the particle's velocity (the speed remains constant), but to change the *direction* of the velocity. In general, forces in the direction of motion change the speed, while forces at right angles to the direction of motion change the direction.

In Exercise 2, a single force caused the circular motion. In order to solve more complicated problems involving circular motion, we need to be able to resolve forces radially and tangentially. For this we need appropriate

unit vectors. The radial and tangential unit vectors are introduced in Subsection 1.3, but first we revise the differentiation of vector functions, which was introduced in Unit 3.

1.2 Differentiation of vector functions

In Unit 3 you saw that if $\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$ is a vector function of t, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the (constant) Cartesian unit vectors, then

$$\dot{\mathbf{f}}(t) = \dot{f}_1(t)\,\mathbf{i} + \dot{f}_2(t)\,\mathbf{j} + \dot{f}_3(t)\,\mathbf{k}. \tag{4}$$

We can use this result to obtain rules for differentiating sums and products of vectors. For example, if the scalar h and the vector $\mathbf{f}$ are both functions of the scalar variable t, then we can show that

$$\frac{d}{dt}(h\mathbf{f}) = \frac{dh}{dt}\,\mathbf{f} + h\,\frac{d\mathbf{f}}{dt}.$$

For ease of reading, we frequently write $\mathbf{f}$, f_i, etc., rather than $\mathbf{f}(t)$, $f_i(t)$, etc., i.e. omitting explicit reference to the variable t.

This follows by first writing the vector function $\mathbf{f}$ in Cartesian component form as

$$\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k},$$

so

$$h\mathbf{f} = (hf_1)\mathbf{i} + (hf_2)\mathbf{j} + (hf_3)\mathbf{k}.$$

To differentiate $h\mathbf{f}$, we apply equation (4) and differentiate the three components hf_1, hf_2 and hf_3. These are products of scalar functions, so we can use the rule for differentiating the product of two scalar functions, giving

$$\frac{d}{dt}(hf_i) = \frac{dh}{dt}f_i + h\,\frac{df_i}{dt} \quad (i = 1, 2, 3).$$

Therefore

$$\begin{aligned}\frac{d}{dt}(h\mathbf{f}) &= \left(\frac{dh}{dt}f_1 + h\,\frac{df_1}{dt}\right)\mathbf{i} + \left(\frac{dh}{dt}f_2 + h\,\frac{df_2}{dt}\right)\mathbf{j} + \left(\frac{dh}{dt}f_3 + h\,\frac{df_3}{dt}\right)\mathbf{k} \\ &= \frac{dh}{dt}\,(f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}) + h\left(\frac{df_1}{dt}\mathbf{i} + \frac{df_2}{dt}\mathbf{j} + \frac{df_3}{dt}\mathbf{k}\right) \\ &= \frac{dh}{dt}\,\mathbf{f} + h\,\frac{d\mathbf{f}}{dt},\end{aligned}$$

which is the required result.

Exercise 3

If the vectors $\mathbf{f}$ and $\mathbf{g}$ are both functions of the scalar variable t, show that

$$\frac{d}{dt}(\mathbf{f}\cdot\mathbf{g}) = \frac{d\mathbf{f}}{dt}\cdot\mathbf{g} + \mathbf{f}\cdot\frac{d\mathbf{g}}{dt}.$$

We can similarly derive the rules for differentiating the other possible combinations of scalar and vector functions.

Rules for differentiating vector functions

If the scalar h and the vectors $\mathbf{f}$ and $\mathbf{g}$ are functions of the scalar variable t, then

$$\frac{d}{dt}(\mathbf{f}+\mathbf{g}) = \frac{d\mathbf{f}}{dt} + \frac{d\mathbf{g}}{dt}, \tag{5}$$

$$\frac{d}{dt}(h\mathbf{f}) = \frac{dh}{dt}\,\mathbf{f} + h\,\frac{d\mathbf{f}}{dt}, \tag{6}$$

$$\frac{d}{dt}(\mathbf{f}\cdot\mathbf{g}) = \frac{d\mathbf{f}}{dt}\cdot\mathbf{g} + \mathbf{f}\cdot\frac{d\mathbf{g}}{dt}, \tag{7}$$

$$\frac{d}{dt}(\mathbf{f}\times\mathbf{g}) = \frac{d\mathbf{f}}{dt}\times\mathbf{g} + \mathbf{f}\times\frac{d\mathbf{g}}{dt}. \tag{8}$$

The rules for differentiating products bear a strong resemblance to the rule for differentiating the product of two scalar functions. Note that in the rule for differentiating a cross product, the order of the factors in each term is important.

These rules will prove useful in the next subsection.

Exercise 4

Differentiate $\mathbf{f}\cdot\mathbf{f}$, where $\mathbf{f}$ is a vector function of the scalar variable t.

In Exercise 4 you showed that

$$\frac{d}{dt}(\mathbf{f}\cdot\mathbf{f}) = 2\mathbf{f}\cdot\frac{d\mathbf{f}}{dt}.$$

This leads to an important result for a vector of *constant magnitude*. Suppose that $\mathbf{e}$ is such a vector, so $\mathbf{e}\cdot\mathbf{e} = |\mathbf{e}|^2 = \text{constant}$. Then

$$\frac{d}{dt}(\mathbf{e}\cdot\mathbf{e}) = 0.$$

Using the result of Exercise 4, this leads directly to

$$\mathbf{e}\cdot\frac{d\mathbf{e}}{dt} = 0.$$

Hence if a non-zero vector $\mathbf{e}$ has constant magnitude, then its derivative $d\mathbf{e}/dt$ is always either zero or perpendicular to $\mathbf{e}$. This can be a useful check on the derivatives of unit vectors, as will become clear in the next subsection.

1.3 Circular motion in polar coordinates

If position is given in Cartesian coordinates (x, y), then it is natural to write the position vector $\mathbf{r}$ in terms of the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$, that is,

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}. \tag{9}$$

Then the velocity and acceleration vectors are

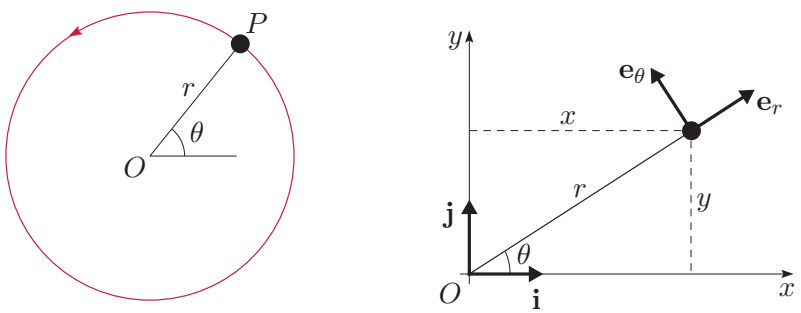
$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{x}(t)\,\mathbf{i} + \dot{y}(t)\,\mathbf{j}, \tag{10}$$

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t) = \ddot{x}(t)\,\mathbf{i} + \ddot{y}(t)\,\mathbf{j}. \tag{11}$$

The ease of derivation of equations (10) and (11) from equation (9) is due to the fact that $\mathbf{i}$ and $\mathbf{j}$ are constant vectors. Only the coordinates x and y, which are functions of time t, vary during the motion.

However, for a particle P moving in a circle (as shown in Figure 4(a)), it is convenient to use polar coordinates (r, θ), with an origin O at the centre of the circle, to describe the position vector $\mathbf{r}$ of the particle. With this choice of coordinate system, the coordinate r is a constant for circular motion (being the radius of the circle), and only the coordinate θ is a function of time t. However, in order to discuss the motion of a particle in a vector context using polar coordinates, we need the two orthogonal unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ (see Figure 4(b)) that were introduced in Unit 15. Recall that $\mathbf{e}_r$ is a unit vector in the direction of increasing r (the **radial direction**), and $\mathbf{e}_\theta$ is a unit vector in the direction of increasing θ (the **tangential direction**).

In contexts where the motion is not circular – and sometimes even when it is – the direction of increasing θ is called the **transverse direction**. For non-circular motion, the transverse vector does not in general coincide with the tangent vector.



(a) (b)

Figure 4 (a) Particle P moving in a circle. (b) Unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ for polar coordinates (r, θ), and their relationship to Cartesian coordinates (x, y) and unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Before we obtain expressions for the position, velocity and acceleration of the particle P in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$, we need to recall how each of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ can be expressed as a linear combination of $\mathbf{i}$ and $\mathbf{j}$. In Unit 15 it was found that

$$\mathbf{e}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}, \tag{12}$$

$$\mathbf{e}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}. \tag{13}$$

Since $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are functions of the polar coordinate angle θ, which is a function of time t, it follows that $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are not constant functions, like $\mathbf{i}$ and $\mathbf{j}$, but are functions of time. However, since their magnitudes are constant (they are unit vectors), it is only their directions that vary with time.

We also need to express the time derivatives $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_\theta$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Differentiating equation (12) with respect to time (since θ is a function of time) using the chain rule, we obtain

$$\dot{\mathbf{e}}_r = (-\sin\theta)\dot{\theta}\mathbf{i} + (\cos\theta)\dot{\theta}\mathbf{j} = \dot{\theta}((-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}) = \dot{\theta}\mathbf{e}_\theta,$$

where we have used the expression for $\mathbf{e}_\theta$ given by equation (13).

Recall that the chain rule is

$$\frac{df(\theta(t))}{dt} = \frac{df}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{df}{d\theta}$$

Exercise 5

By differentiating equation (13) with respect to time, and using equation (12), show that

$$\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r.$$

Since $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are orthogonal, $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$ is perpendicular to $\mathbf{e}_r$, and in addition, $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$ is perpendicular to $\mathbf{e}_\theta$. These are special cases of the result proved after Exercise 4 that the derivative of a non-zero vector of constant magnitude is always either zero or perpendicular to that vector.

Polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$

Given a polar coordinate system and a two-dimensional Cartesian coordinate system with the same origin O, where the positive x-axis corresponds to the axis from which the polar coordinate θ is measured, the polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are related to the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ by

$$\mathbf{e}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}, \tag{14}$$

$$\mathbf{e}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}. \tag{15}$$

The derivatives with respect to time of the polar unit vectors are

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta, \tag{16}$$

$$\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r. \tag{17}$$

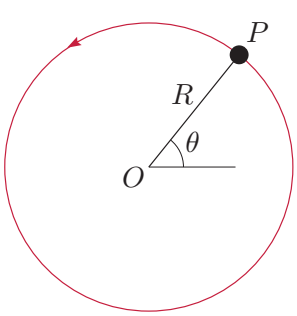


Figure 5 Particle P moving in a circle of radius R

We are now in a position to obtain expressions for the position $\mathbf{r}$, the velocity $\mathbf{v} = \dot{\mathbf{r}}$ and the acceleration $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$ of a particle P moving in a circle of radius R, say (see Figure 5), in terms of the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. For any point on the circle $r = R$, the position vector of P is in the direction of $\mathbf{e}_r$ thus can be expressed as

$$\mathbf{r} = R\mathbf{e}_r.$$

Differentiating this equation with respect to time, and remembering that R is a constant, we have

$$\dot{\mathbf{r}} = R\dot{\mathbf{e}}_r.$$

Using equation (16), this may be written as

$$\dot{\mathbf{r}} = R\dot{\theta}\mathbf{e}_\theta. \tag{18}$$

Exercise 6

By differentiating equation (18) with respect to time, remembering that θ and $\mathbf{e}_\theta$ are functions of time but that R is a constant, show that the acceleration of a particle moving in a circle of radius R is

$$\ddot{\mathbf{r}} = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta.$$

It is useful to collect together the results that apply whenever the motion is circular (for both uniform circular motion, considered in Section 2, and non-uniform circular motion, considered in Section 3).

Position, velocity and acceleration in polar coordinates for circular motion

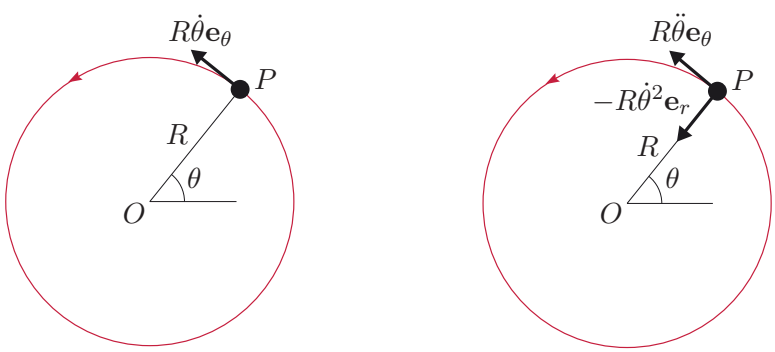
In polar coordinates, the position, velocity and acceleration of a particle moving in the circle $r = R$ are

$$\mathbf{r} = R\mathbf{e}_r, \tag{19}$$

$$\dot{\mathbf{r}} = R\dot{\theta}\mathbf{e}_\theta, \tag{20}$$

$$\ddot{\mathbf{r}} = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta. \tag{21}$$

It is geometrically clear (see Figure 6(a)) that the velocity of a particle moving in a circle is tangential to the circle, and equation (20) confirms this. On the other hand (see Figure 6(b)), the acceleration has a radial component $-R\dot{\theta}^2\mathbf{e}_r$ (arising from the rate of change of the direction of the velocity) as well as a tangential component $R\ddot{\theta}\mathbf{e}_\theta$ (arising from the rate of change of the speed). Furthermore, the component of the acceleration in the radial direction is *negative*, which indicates that the acceleration of a particle moving in a circle has a component of magnitude $R\dot{\theta}^2$ directed *inwards*, towards the centre of the circle. This inward component is commonly called the **centripetal acceleration**, but you must remember that the acceleration for circular motion in general also has a non-zero tangential component.



(a) (b)

Figure 6 Particle P moving in a circle of radius R, showing (a) its velocity, and (b) its acceleration

Results (19) to (21) are the generalisation of the results of Subsection 1.1 to motion where the angular speed is not constant. If the angular speed is constant (i.e. the circular motion is *uniform*), then $\dot{\theta} =$ constant and $\ddot{\theta} = 0$, so equation (21) reduces to $\ddot{\mathbf{r}} = -R\dot{\theta}^2\mathbf{e}_r$ and the acceleration is solely towards the centre of the circle, as you saw in Subsection 1.1.

Exercise 7

A particle is moving on a unit circle centred on the origin O, that is, on the circle $r = 1$. Its polar coordinate angle θ at time t is given by $\theta(t) = t^2$.

Find the velocity and acceleration of the particle at time t in terms of the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

2 Uniform circular motion

The previous section derived formulas that apply whenever an object is moving in a circle. This section considers the special case where the angular rate of rotation is constant, that is, **uniform circular motion**. Non-uniform circular motion is studied in Section 3.

2.1 Introductory examples

We begin with a summary of results from Section 1 specialised to the case of uniform circular motion, that is, the case where $\ddot{\theta} = 0$.

Kinematics of uniform circular motion

Consider a particle moving in a circle of radius R, with centre at the origin O, having a constant rate of rotation $\dot{\theta}$ and hence a constant angular speed $\omega = |\dot{\theta}|$.

- The velocity of the particle is

$$\dot{\mathbf{r}} = R\dot{\theta}\mathbf{e}_\theta, \tag{22}$$

  i.e. the velocity of the particle is purely tangential and has constant magnitude $v = |\dot{\mathbf{r}}| = R|\dot{\theta}| = R\omega$.

- The acceleration of the particle is

$$\ddot{\mathbf{r}} = -R\dot{\theta}^2\mathbf{e}_r = -R\omega^2\mathbf{e}_r = -\frac{v^2}{R}\,\mathbf{e}_r, \tag{23}$$

  i.e. the acceleration is directed towards the centre of the circle and has constant magnitude $|\ddot{\mathbf{r}}| = R\dot{\theta}^2 = R\omega^2 = v^2/R$.

- The time taken for one complete revolution of the circle is

$$T = \frac{2\pi}{\omega}. \tag{24}$$

This follows from ωt being the magnitude of the angle measured from the x-axis at time t, and the period being defined as the time for this angle to equal 2π.

Exercise 8

A fly sits at the tip of the minute hand of a clock. The hand has length 0.5 m and can be assumed to move with constant angular speed.

Find the angular speed, the speed and the magnitude of the acceleration of the fly.

If there are no forces acting on a particle, then by Newton's first law, it will either remain at rest or move with constant speed in a straight line. Therefore if the particle moves on a circular path, there must be some force or forces causing it to do so, even when the circular motion is uniform. In Subsection 1.1 you saw that a particle moving on a circle of radius R with constant angular speed ω and constant speed v is being accelerated towards the centre of the circle with an acceleration of magnitude $R\omega^2 = v^2/R$. If the particle has mass m, then Newton's second law states that there must be a force

$$\mathbf{F} = m\ddot{\mathbf{r}} = -mR\omega^2\mathbf{e}_r = -\frac{mv^2}{R}\,\mathbf{e}_r$$

acting on the particle. This force is directed towards the centre of the circle and has constant magnitude. There is no tangential component of the acceleration, so there is no component of the total force (i.e. resultant force) acting on the particle in this direction.

Similarly, there is no component of the resultant force acting at right angles to the plane of the motion. In order to be able to use this fact to solve problems, we need to introduce a third unit vector $\mathbf{k}$, which is perpendicular to the plane of motion and whose direction is chosen so that the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{k}$ form a right-handed system. An example of such a right-handed system is shown in Figure 7.

You met the term right-handed system in Unit 2.

Example 1

Two particles A and B, of equal mass m, are connected by a model string that passes through a small smooth hole in a smooth horizontal table. Particle A is on the surface of the table, while particle B is suspended beneath the table. Particle A describes circles of radius R with constant angular speed ω, as a result of which particle B is static.

Find the angular speed of particle A.

Solution

Without loss of generality, we can assume that particle A moves in an anticlockwise direction, when viewed from above the table. We will describe the motion of particle A using a polar coordinate system in the plane of the table surface with origin at O, where O is the centre of the circle described by particle A.

The situation, together with the force diagrams for the two particles, is shown in Figure 7. $\mathbf{T}_1$ and $\mathbf{T}_2$ are the tension forces due to the string on particles A and B, respectively, $\mathbf{W}_1$ and $\mathbf{W}_2$ are the weights of particles A and B, respectively, and $\mathbf{N}$ is the normal reaction of the table on particle A (there is no friction force since the table is smooth).

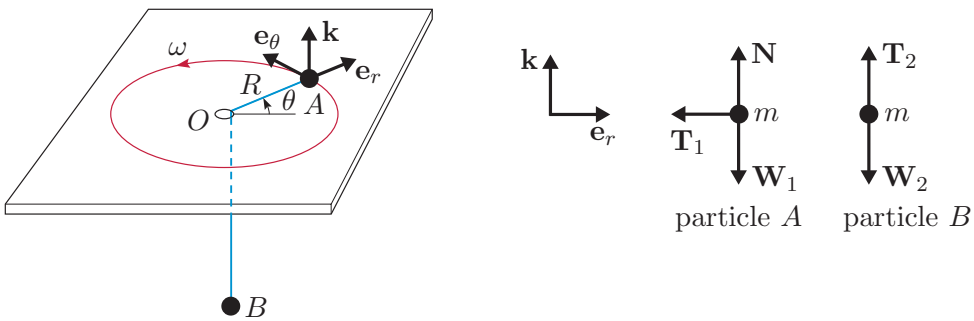


Figure 7 Two particles, A and B, connected by a string passing through the centre of a circle described by A

First consider particle B. There are two forces acting on this particle: the tension force $\mathbf{T}_2 = |\mathbf{T}_2|\mathbf{k}$ due to the string, and its weight $\mathbf{W}_2 = -mg\mathbf{k}$, where $\mathbf{k}$ is a unit vector pointing vertically upwards. Since particle B is static, the total force acting on it is zero, so

$$\mathbf{T}_2 + \mathbf{W}_2 = \mathbf{0},$$

that is,

$$|\mathbf{T}_2|\mathbf{k} - mg\mathbf{k} = \mathbf{0}.$$

The vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{k}$ are mutually perpendicular, so we can resolve in these directions in the same way as for the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Hence, resolving in the $\mathbf{k}$-direction, we obtain

$$|\mathbf{T}_2| = mg.$$

Now consider particle A. There are three forces acting on this particle: the tension force $\mathbf{T}_1 = -|\mathbf{T}_1|\mathbf{e}_r$ due to the string, the normal reaction $\mathbf{N} = |\mathbf{N}|\mathbf{k}$ of the table on the particle, and its weight $\mathbf{W}_1 = -mg\mathbf{k}$. Applying Newton's second law to particle A gives

$$m\ddot{\mathbf{r}} = \mathbf{T}_1 + \mathbf{N} + \mathbf{W}_1,$$

that is, using equation (23),

$$-mR\omega^2\mathbf{e}_r = -|\mathbf{T}_1|\mathbf{e}_r + |\mathbf{N}|\mathbf{k} - mg\mathbf{k}.$$

Resolving in the direction of $\mathbf{e}_r$ gives

$$|\mathbf{T}_1| = mR\omega^2.$$

Since we have a model string and a smooth hole, $|\mathbf{T}_1| = |\mathbf{T}_2|$ thus $mR\omega^2 = mg$. Hence $\omega^2 = g/R$ and $\omega = \sqrt{g/R}$.

So the angular speed of particle A is $\sqrt{g/R}$.

Exercise 9

A particle P of mass m moves in a circle with constant angular speed ω on a smooth horizontal table. It is attached to an end of a model string of length l. The other end of the string is fixed to a point on the table.

What is the tension in the string?

Exercise 10

A child places a small toy of mass m on a playground roundabout and spins the roundabout so that it describes one complete revolution every two seconds. The toy stays at the same point relative to the roundabout while the latter rotates. The coefficient of static friction between the toy and the roundabout is $\mu = \frac{1}{2}$.

By modelling the toy as a particle and the roundabout as a horizontal disc rotating uniformly about its centre, show that the greatest possible distance of the toy from the centre of the roundabout is about 0.5 m.

(*Hint*: Recall from Unit 2 that the condition that an object on a rough surface does not slip is $|\mathbf{F}| \leq \mu|\mathbf{N}|$, where $\mathbf{F}$ is the friction force and $\mathbf{N}$ is the normal reaction on the object.)

Exercise 11

A particle of mass m is joined by a model spring of stiffness k and natural length l_0 to a fixed point O on a smooth horizontal table. The particle is moving in a circle on the table, the circle being centred on O, with a constant angular speed ω (where $\omega < \sqrt{k/m}$).

Find the length of the model spring in terms of m, k, l_0 and ω.

(*Hint*: Recall Hooke's law from Unit 9, which states that the spring exerts a force $\mathbf{H} = k(l - l_0)\widehat{\mathbf{s}}$ on the particle, where l is the length of the spring and $\widehat{\mathbf{s}}$ is a unit vector from the particle to the centre of the spring.)

2.2 Geostationary satellites

An interesting application of uniform circular motion concerns communications satellites. Some of the most important satellites are those with *geostationary orbits*, that is, satellites that maintain a fixed position relative to the Earth's rotating surface. Signals can be sent to and from such a satellite without the need to adjust continually the orientations of the transmitter and the receiver. In order that the geostationary satellite should maintain its position, it must have a circular orbit in the equatorial plane and an orbital period of one sidereal day, so its angular speed is $\omega = 2\pi/T$, where

$$T = 1 \text{ day} = 23.9343 \times 60 \times 60 \text{ s}.$$

Hence ω is approximately $7.292 \times 10^{-5} \text{ rad s}^{-1}$.

The sidereal day is the time required for a complete rotation of the Earth relative to a particular star. You will examine this in Example 3 in Section 4.

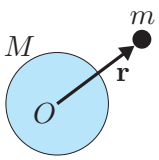


Figure 8 Position vector of a satellite

The force causing the circular motion is the gravitational force of attraction on the satellite due to the presence of the Earth. This force is attractive and directed towards the centre of the Earth. Empirical evidence has shown that it is well modelled by

$$\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^3}\,\mathbf{r}, \tag{25}$$

where m is the mass of the satellite, M is the mass of the Earth, $\mathbf{r}$ is the position vector of the satellite with respect to the centre of the Earth (as shown in Figure 8), and G is a constant, known as the **gravitational constant**, whose value in SI units is $6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$.

Equation (25) is a special case of *Newton's law of universal gravitation*, which models the gravitational force between any two particles. (We are assuming that all other forces can be neglected.)

Newton's law of universal gravitation

The gravitational force of attraction exerted on a particle of mass m_1 by a particle of mass m_2 is

$$\mathbf{F} = -\frac{Gm_1m_2}{|\mathbf{r}|^3}\,\mathbf{r}, \tag{26}$$

where $\mathbf{r}$ is the position vector of the particle of mass m_1, relative to the particle of mass m_2, and $G = 6.674 \times 10^{-11}$ (in $\mathrm{N\,m^2\,kg^{-2}}$) is the gravitational constant.

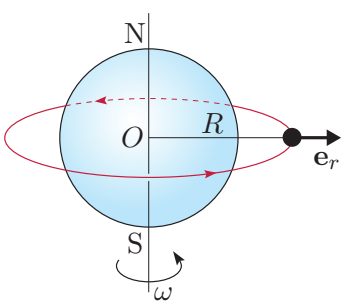


Figure 9 Satellite orbiting the Earth

In order to use Newton's law of universal gravitation to model the gravitational force on a geostationary satellite, we assume that we can model the Earth and the satellite as particles (evidence shows that this assumption is reasonable). Using polar coordinates in the equatorial plane, with origin O at the centre of the Earth, the position vector of the satellite can be written as $\mathbf{r} = R\mathbf{e}_r$, where R is the radius of the orbit (see Figure 9). So equation (25) can be written as

$$\mathbf{F} = -\frac{GmM}{R^2}\,\mathbf{e}_r.$$

Applying Newton's second law and equation (23) to the satellite, we obtain

$$-mR\omega^2\mathbf{e}_r = -\frac{GmM}{R^2}\,\mathbf{e}_r,$$

which leads to

$$R = \left(\frac{GM}{\omega^2}\right)^{1/3}.$$

Now the mass M of the Earth is $5.972 \times 10^{24}\,\mathrm{kg}$. Substituting the values of G, M and ω into the above equation, we obtain (in m)

$$R \simeq 4.216 \times 10^7.$$

The equatorial radius of the Earth is 6378.2 km, so all geostationary satellites must orbit at approximately 35 800 km above the Earth's surface.

Exercise 12

A good approximation for planetary motion is to assume that a planet orbits in uniform circular motion around the Sun under the action of their mutual gravitational attraction. Model the Sun and the planet as particles, take the origin at the centre of mass of the Sun, and take R to be the radius of the planet's orbit around this origin, so the position of the planet is $\mathbf{r} = R\mathbf{e}_r$. Newton's law of universal gravitation then tells us that a good model of the gravitational force on the planet is

$$\mathbf{F} = -\frac{GmM}{R^2}\,\mathbf{e}_r,$$

where m is the mass of the planet, and M is the mass of the Sun.

Verify that the square of the period T of the orbit is proportional to the cube of the radius of the orbit, that is,

$$T^2 \propto R^3.$$

This is Kepler's third law of planetary motion applied to circular orbits.

2.3 Conical pendulums

So far in this section we have considered examples of motion where the force causing the circular motion is in the plane of the motion. In Exercises 13–15 you are asked to consider examples where this is not the case.

Exercise 13

A *conical pendulum* is a pendulum whose bob traces out a horizontal circle rather than a vertical circle. The pendulum can be modelled as a particle of mass m joined by a light model rod of length l to a fixed point (the pivot). As the particle traces out a horizontal circle below the fixed point, the rod traces out the curved surface of a cone whose semi-vertical angle α is the inclination of the rod to the vertical (see Figure 10). Assume that for any given angle α, the bob performs uniform circular motion with angular speed ω.

(a) Determine the tension in the rod in terms of m, α and the magnitude of the acceleration due to gravity, g.

(b) Show that such motion is possible only if $\omega > \omega_0$, where the minimum angular speed is given by $\omega_0 = \sqrt{g/l}$, and that $0 < \alpha < \pi/2$. Show also that $\alpha \to \pi/2$ as $\omega \to \infty$.

(c) Show that for a given angular speed $\omega > \omega_0$, the vertical distance of the bob below the pivot is independent of l. In other words, regardless of the length of the rod, and with a given angular speed, the bob will always rotate at the same vertical distance below the pivot.

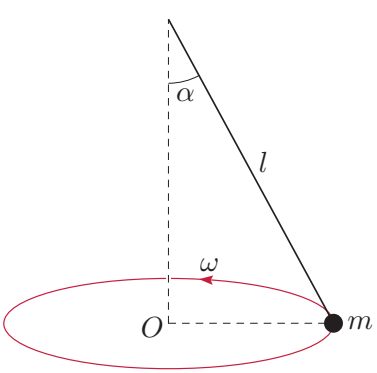


Figure 10 Conical pendulum

You saw in Exercise 13 that the angular speed of a conical pendulum is given by

$$\omega = \sqrt{g/(l\cos\alpha)},$$

where l is the length of the pendulum rod and α is the angle that the rod makes with the vertical. For a fixed l, this means that ω depends solely on α. As α increases, so does ω (since $\cos\alpha$ decreases), and $\omega \to \infty$ as $\alpha \to \frac{\pi}{2}$.

The fact that the angular speed of a conical pendulum depends on the angle made with the vertical is used in the design of a *governor* of a steam engine (see Figure 11), which regulates the supply of steam to the engine and thereby controls the speed at which the engine operates.

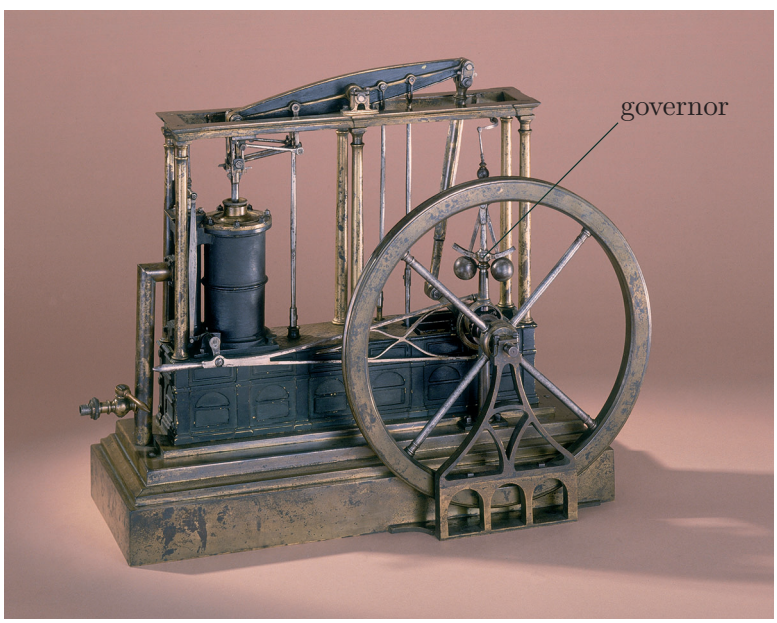


Figure 11 Steam engine with governor indicated

A model of a governor is illustrated in Figure 12.

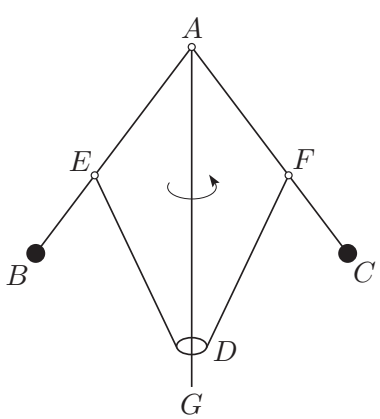


Figure 12 Model of a governor

The model consists of two light rods AB and AC, freely hinged at A to a vertical shaft AG that is rotated by the engine. At the other ends of the rods are two equal weights at B and C. Two other light rods DE and DF are freely hinged to the rods AB and AC, respectively, and to a collar at D that can slide freely up and down the shaft AG. A lever is attached to the collar at D and operates a valve that admits steam to the engine – as the collar rises, the amount of steam is decreased.

So when the angular speed of the shaft increases as more steam enters the engine, the weights at B and C rise and pull the collar at D up the shaft. This shuts off some of the steam entering the engine, and the speed of rotation is decreased. Similarly, when the speed of rotation decreases, the collar at D falls and more steam is admitted to the engine.

Exercise 14

A model of a steam governor assumes that in Figure 12, the points B and E coincide, and the points C and F coincide, that is, the rods DE and DF are freely hinged at B and C, respectively. From the symmetry of Figure 12, we need model only half of the governor, that is, two rods and one weight, and one half-collar at D. We model the rods and the shaft as light model rods, the weight as a particle P, and the half-collar as another particle. The model is shown in Figure 13.

Assume that the half-collar particle remains in smooth contact with the shaft, so there is no friction and the only force on the half-collar particle due to the shaft is the normal reaction. Also assume that there are no friction forces (or resistive torques) at any of the hinges, that both particles have the same mass m, and that both rods have the same length l.

The particle P rotates about the shaft in a horizontal circle with constant angular speed ω. Because both rods have the same length, each rod makes the same angle α with the vertical.

(a) Determine the tensions in the rods.

(*Hint*: For fixed values of ω and α, the half-collar particle at D does not move.)

(b) Find an expression for the angular speed ω in terms of g, l and α, and find an expression for the angle α of inclination in terms of g, l and ω.

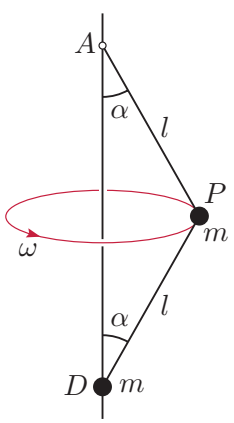


Figure 13 Simplified model of a governor

Exercise 15

A model string ACB has a particle of mass m attached at one end A and a particle of mass $2m$ attached at the other end B. The string is threaded through a smooth fixed ring C. The particle B is at rest vertically below the ring C, and the particle A is rotating in a horizontal circle of radius R with constant angular speed ω, as shown in Figure 14.

(a) Find the angle α of inclination of AC to the vertical.

(b) Find, in terms of R and g, the angular speed ω of particle A.

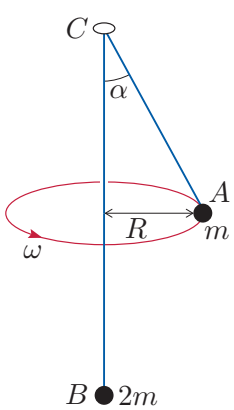


Figure 14

2.4 Cornering

In this subsection we look at the application of the formulas for circular motion to the real-world problem in the following example.

Example 2

Consider a vehicle of mass m moving at constant speed v around a circular bend of radius R. Suppose that the road has a horizontal surface, that is, it is not banked or cambered.

It may seem odd to use the coefficient of *static* friction here. However, there is no *sideways* motion between the tyres and the road.

(a) If the coefficient of static friction between the vehicle and the road is μ, what is the maximum constant speed v at which the vehicle can go around the corner without slipping?

(b) Suppose that the vehicle is driven around a roundabout on a circle of radius 15 m at a constant speed of $10\,\mathrm{m\,s^{-1}}$ without skidding. What does this imply about the coefficient of static friction?

Solution

(a) Model the vehicle as a particle, at position X. Choose the $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{k}$ coordinate system shown in Figure 15.

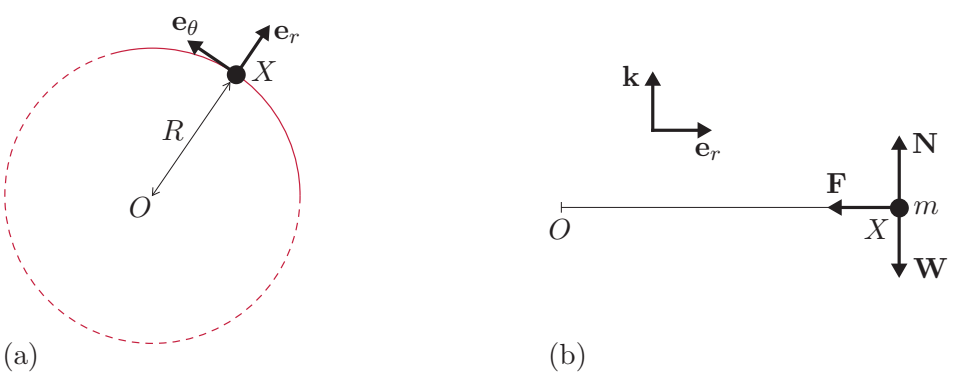


Figure 15 Vehicle following a circular bend at constant speed: (a) viewed from above; (b) viewed from the side, showing a vertical plane through OX and the forces acting in this plane

The forces acting on the vehicle are not all in one plane, so it is hard to draw a complete force diagram. Figure 15(b) shows the forces in a vertical plane through the vehicle and the centre of the circle, which are the weight $\mathbf{W}$, the normal reaction $\mathbf{N}$, and the sideways friction force $\mathbf{F}$. There are other forces acting on the vehicle, such as air resistance and the force propelling the vehicle. For the purposes of this model, we lump them together as one force $\mathbf{C}$, whose magnitude is unknown but whose direction is along the direction of travel, that is, tangential to the circle and perpendicular to the plane shown in Figure 15(b).

The vehicle is being treated as a particle, so $\mathbf{W}$ is the weight of the vehicle together with its occupants. The normal reaction $\mathbf{N}$ is the resultant of the separate forces acting at each tyre/road contact.

Applying Newton's second law to the vehicle gives

$$\mathbf{N} + \mathbf{W} + \mathbf{F} + \mathbf{C} = m\mathbf{a}. \tag{27}$$

Now we model the forces that appear in this equation. By inspection of Figure 15(b), we have

$$\mathbf{W} = -mg\mathbf{k}, \quad \mathbf{N} = |\mathbf{N}|\mathbf{k}, \quad \mathbf{F} = -|\mathbf{F}|\mathbf{e}_r.$$

By definition, the force $\mathbf{C}$ is tangential to the circle, so

$$\mathbf{C} = |\mathbf{C}|\mathbf{e}_\theta.$$

Since the vehicle is moving in a circular path at constant speed, the acceleration is $\mathbf{a} = (-v^2/R)\mathbf{e}_r$, using equation (23). Now we can resolve equation (27) in each of three orthogonal directions.

Resolving equation (27) in the $\mathbf{e}_\theta$-direction gives

$$|\mathbf{C}| = 0, \tag{28}$$

resolving equation (27) in the $\mathbf{k}$-direction gives

$$|\mathbf{N}| - mg = 0, \tag{29}$$

and resolving equation (27) in the $\mathbf{e}_r$-direction gives

$$-|\mathbf{F}| = -\frac{mv^2}{R}. \tag{30}$$

From equation (28) we see that the forces tangential to the circle must balance, since we have uniform circular motion. This justifies the modelling assumption that we can lump them all together.

From equation (29) we obtain $|\mathbf{N}| = mg$, so by the force model for static friction, we have

$$|\mathbf{F}| \leq \mu|\mathbf{N}| = \mu mg.$$

But from equation (30) we obtain $|\mathbf{F}| = mv^2/R$, which we can substitute into this inequality to obtain

$$\frac{mv^2}{R} \leq \mu mg,$$

that is,

$$v^2 \leq \mu Rg. \tag{31}$$

So the maximum (constant) speed at which the circular bend can be negotiated without skidding is $v = \sqrt{\mu Rg}$.

(b) We have $v = 10$ and $R = 15$. So from expression (31), we require that

$$\mu \geq \frac{v^2}{Rg} = \frac{100}{15 \times 9.81} \simeq 0.68.$$

So the coefficient of static friction must be at least 0.68 to avoid skidding.

Exercise 16

Suppose that the coefficient of static friction between a particular car and the road in dry conditions is 1.3, and assume that the car follows a circular path at constant speed.

(a) At what speed could the car be driven around a roundabout on a circular path of radius 10 m in dry conditions without skidding?

(b) What would be the smallest radius of the roundabout that would enable the car to drive around it in dry conditions at $15\,\mathrm{m\,s^{-1}}$?

(c) Suppose that when conditions are wet, the coefficient of static friction between the car and the road is halved. What would be the smallest radius of the roundabout that would enable the car to be driven around it in wet conditions at $15\,\mathrm{m\,s^{-1}}$?

2.5 Chair-o-planes

A chair-o-plane, as mentioned in the Introduction (see Figure 1), relies on circular motion to supply its thrills. In this subsection we examine its motion by first working through an exercise.

Exercise 17

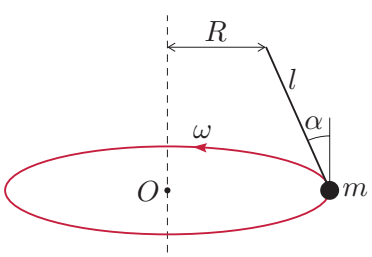


Figure 16 Model of a chair-o-plane roundabout

A chair-o-plane roundabout consists of seats that are suspended from chains. As the roundabout rotates, the chairs swing out at an angle. We will model a seat and passenger as a particle of mass m suspended from a model string of length l. The other end of the string is fixed to a point a horizontal distance R from the vertical axis of rotation, as shown in Figure 16. We assume that the roundabout is rotating at constant angular speed ω and that the system has reached a steady state with the string inclined to the vertical at an angle α.

Determine the equation of motion of the system, and hence show that

$$\frac{g}{\omega^2}\tan\alpha = R + l\sin\alpha. \tag{32}$$

It is not easy to solve equation (32) for α, but this is not necessary in order to answer the questions posed in the Introduction.

Question 1 Will a child swing out at a greater angle than a heavier adult?

Question 2 Will the people on the outside swing out at a greater angle than those on the inside? (In terms of the parameters in equation (32), this relates to how α varies with R, all other parameters being fixed.)

Question 3 How does the angle at which people swing out depend on the speed of rotation of the roundabout?

You may like to stop at this point and try to answer these questions yourself.

The answer to Question 1 is that the child and the adult will swing outwards at the same angle, because equation (32) is independent of m, the mass of the particle.

To answer Questions 2 and 3, the arguments can be made easier by looking at the graphs shown in Figure 17. Plotted on the same set of axes are the graphs of $y = (g/\omega^2)\tan\alpha$ and $y = R + l\sin\alpha$ (the two sides of equation (32)), for fixed ω, R and l, and for $0 \leq \alpha \leq \frac{\pi}{2}$. From equation (32), the intersection of these two graphs gives the angle α of inclination of the chain to the vertical.

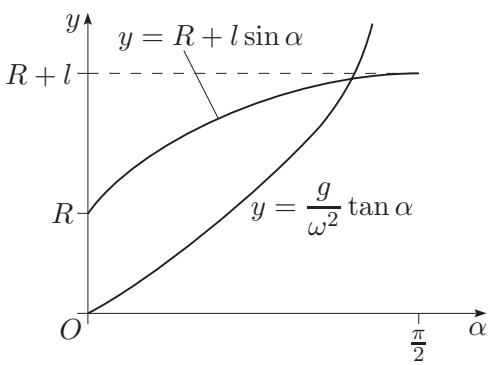


Figure 17 Graphical solution of $(g/\omega^2)\tan\alpha = R + l\sin\alpha$, at the intersection point of the two curves

The first point to note is that for any positive values of R, l and ω, the graphs intersect once for $0 \leq \alpha < \frac{\pi}{2}$, so the motion is possible for all angular speeds ω.

To answer Question 2, note from Figure 17 that as R increases, the graph of $y = R + l\sin\alpha$ is translated upwards (assuming that all other parameters are fixed). This moves the point of intersection of the two graphs to the right. So a larger value of R corresponds to an angle α closer to $\alpha = \pi/2$. The answer to Question 2 is therefore that people on the outside do swing out further than those on the inside.

To answer Question 3, consider what happens to the point of intersection of the two graphs as ω increases. In this case, g/ω^2 decreases, and for a given α, the curve $y = (g/\omega^2)\tan\alpha$ becomes less steep. This has the effect of moving the point of intersection of the graphs to the right and hence increases the angle α of inclination. So the answer to Question 3 is that increasing the angular speed ω of the roundabout increases the angle of inclination α: as ω tends to infinity, the angle α tends to $\pi/2$, that is, the chains are almost horizontal if the roundabout has a very large angular speed.

(It can also be deduced from the graphs that the angle α of inclination increases as the length l of the chains increases.)

3 Non-uniform circular motion

In this section we look at **non-uniform circular motion**, that is, circular motion in which the angular rate of rotation is not constant. We will continue to make use of the polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Most of the section is concerned with the motion of a simple pendulum, which you examined in Unit 11. In that unit you modelled the motion using a second-order differential equation obtained by examining the mechanical energy of the system. You have modelled the motion using Newton's second law and Cartesian unit vectors. Here we use Newton's second law and polar unit vectors to obtain an equation for the varying angular speed of the pendulum.

We model the pendulum as a particle P of mass m attached to a light model rod of length l. The other end of the rod is attached to a fixed point O (a model pivot), and the particle moves in a vertical circle about this fixed point, as shown in Figure 18. The angle θ made by the rod with the vertical is measured anticlockwise from the downward vertical. We want to find expressions for the rate of rotation $\dot{\theta}$ and the tension in the rod in terms of θ.

We will use a polar coordinate system in the vertical plane of motion of the pendulum, with origin at the fixed point O, the centre of the circular motion. The forces on the particle are its weight $\mathbf{W}$ and the tension $\mathbf{T}$ due to the rod. The polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$, and a force diagram for the system, are shown in Figure 18.

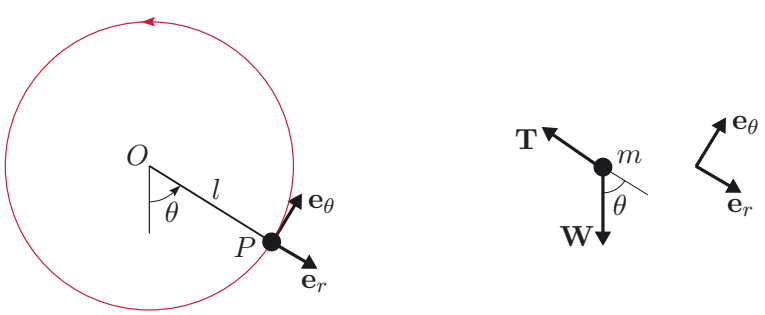


Figure 18 Vertical pendulum pivoted at O with a bob at P

With respect to this coordinate system,

$$\mathbf{W} = (mg\cos\theta)\mathbf{e}_r - (mg\sin\theta)\mathbf{e}_\theta, \tag{33}$$

$$\mathbf{T} = -T\mathbf{e}_r. \tag{34}$$

Note that we cannot write $\mathbf{T} = -|\mathbf{T}|\mathbf{e}_r$, as is often done elsewhere in this module, since the direction of $\mathbf{T}$ can point either towards or away from O depending on whether the rod is under tension or under compression. Instead, equation (34) uses T, which is defined as

$$T = \begin{cases} |\mathbf{T}| & \text{when the rod is under tension,} \\ -|\mathbf{T}| & \text{when the rod is under compression.} \end{cases}$$

Recall from Unit 2 that the modulus of a vector is a non-negative scalar, so $|\mathbf{T}| \geq 0$.

So if $T > 0$, then the force $\mathbf{T}$ is directed towards O so that the rod is under tension, as would be the case when the pendulum is hanging vertically downwards. On the other hand, if $T < 0$, then $\mathbf{T}$ is directed in the opposite direction, away from O so that the rod is under compression and exerts a normal reaction on the bob. Of course, in neither case does the rod's length change. The point where T vanishes can play an important role in further applications, as you will see later in this section.

The acceleration of the particle is non-zero and is given by equation (21):

$$\ddot{\mathbf{r}} = -l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta. \tag{35}$$

Newton's second law applied to the particle gives

$$m\ddot{\mathbf{r}} = \mathbf{W} + \mathbf{T},$$

so, substituting from equations (33), (34) and (35), we obtain

$$-ml\dot{\theta}^2\mathbf{e}_r + ml\ddot{\theta}\mathbf{e}_\theta = (mg\cos\theta)\mathbf{e}_r - (mg\sin\theta)\mathbf{e}_\theta - T\mathbf{e}_r.$$

Resolving in the $\mathbf{e}_r$-direction gives

$$-ml\dot{\theta}^2 = mg\cos\theta - T. \tag{36}$$

Resolving in the $\mathbf{e}_\theta$-direction gives

$$ml\ddot{\theta} = -mg\sin\theta. \tag{37}$$

We concentrate first on integrating equation (37) to obtain an expression for the rate of rotation $\dot{\theta}$. By dividing equation (37) by ml it can be simplified to

$$\ddot{\theta} = -\frac{g}{l}\sin\theta, \tag{38}$$

which is an equation of the form

$$\ddot{\theta} = f(\theta) \tag{39}$$

(where in this case, $f(\theta) = -(g/l)\sin\theta$).

We then multiply equation (39) by $\dot{\theta}$ to obtain

$$\dot{\theta}\ddot{\theta} = \dot{\theta}\, f(\theta), \tag{40}$$

and apply the chain rule to give

$$\frac{d}{dt}(\dot{\theta}^2) = \frac{d(\dot{\theta}^2)}{d\dot{\theta}}\frac{d\dot{\theta}}{dt} = 2\dot{\theta}\ddot{\theta}.$$

Using this result after multiplying equation (40) by 2 leads to

$$\frac{d}{dt}(\dot{\theta}^2) = 2\dot{\theta}\, f(\theta),$$

which can be integrated to yield

$$\dot{\theta}^2 = 2\int f(\theta)\,\dot{\theta}\,dt = 2\int f(\theta)\,\frac{d\theta}{dt}\,dt = 2\int f(\theta)\,d\theta.$$

So with $f(\theta) = -(g/l)\sin\theta$, we arrive at the desired expression for $\dot{\theta}^2$, namely

$$\dot{\theta}^2 = 2\int f(\theta)\,d\theta = -\frac{2g}{l}\int \sin\theta\,d\theta = \frac{2g}{l}\cos\theta + C.$$

The constant of integration C can be found, for example, in terms of the rate of rotation $\dot{\theta}_0$ of the particle at the lowest point of the path, where $\theta = 0$. This initial condition leads to

$$C = \dot{\theta}_0^2 - \frac{2g}{l},$$

so

$$\dot{\theta}^2 = \dot{\theta}_0^2 - \frac{2g}{l}(1 - \cos\theta).$$

When $\theta = 0$, we have $v_0 = l|\dot{\theta}_0|$, so this equation becomes

$$\dot{\theta}^2 = \frac{v_0^2}{l^2} - \frac{2g}{l}(1 - \cos\theta). \tag{41}$$

In theory, this equation can now be integrated to find θ as a function of time t, but in practice, the integral involved cannot be evaluated in terms of elementary functions. However, equation (41) does permit us to calculate the speed $v = l|\dot{\theta}|$ of the particle at any point during its motion.

Equation (41) can also be derived more directly using the law of *conservation of energy*, without the need to resolve forces, let alone solve equation (38). This can be done by applying methods described in Units 9 and 16. Since gravity is a conservative force, the total mechanical energy of the pendulum is constant throughout its motion, and is given by

$$E = \tfrac{1}{2}mv^2 + U(\theta),$$

where $U(\theta)$ is the gravitational potential energy of the bob. This is given by $U = mgh$, where h is the height of the bob above the datum (see Figure 19 and Unit 9, Subsection 3.2). Taking the datum O to be the lowest point of the motion, that is, where $\theta = 0$, we have that $h = l - l\cos\theta$ and hence

$$E = \tfrac{1}{2}mv^2 + mgl(1 - \cos\theta). \tag{42}$$

Since E stays constant throughout the motion, its value can be determined by the condition that $v = v_0$ at $\theta = 0$, which implies that $E = \frac{1}{2}mv_0^2$. Substituting this into equation (42), together with $v^2 = l^2\dot{\theta}^2$, finally yields

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\theta),$$

which can be rearranged to give equation (41).

In addition to gravity there is also the tension **T**, but since **T** is always perpendicular to the motion, it does not contribute to the work done on the particle or therefore to the energy (see Unit 16).

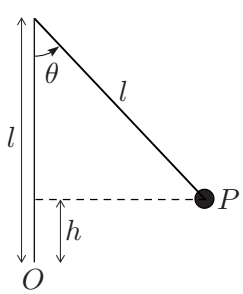


Figure 19 Vertical pendulum of length l with bob at P and datum at O

Exercise 18

Find the value of v_0 for which the bob just comes to rest at $\theta = \pi$, that is, at the highest point possible.

The solution to Exercise 18 implies that if $0 < v_0 < 2\sqrt{gl}$, then the bob oscillates back and forth with an amplitude of less than π, whereas for $v_0 > 2\sqrt{gl}$, the bob rotates in complete circles.

Let us now return to equation (36) and rewrite it as

$$T = mg\cos\theta + ml\dot{\theta}^2, \tag{43}$$

from which you can see that $T > 0$ (i.e. the rod is under tension) for all $|\theta| < \pi/2$ (i.e. whenever the bob is below the pivot), since $\cos\theta > 0$ on this interval, while $\dot{\theta}^2 \geq 0$ for any θ. We can use equation (41) to substitute for $\dot{\theta}^2$ in equation (43), to find the tension in the rod in terms of θ. Thus we have

$$T = mg\cos\theta + \frac{mv_0^2}{l} - 2mg(1 - \cos\theta),$$

so

$$T = \frac{mv_0^2}{l} + mg(3\cos\theta - 2). \tag{44}$$

Equation (44) can be used to determine the sign of T, but note that both $\dot{\theta}^2$ and T increase with increasing v_0.

Let us now consider a slightly different pendulum, where the light model rod is replaced with a model string.

Exercise 19

Consider the motion of a pendulum moving in a vertical plane, whose bob is attached to a fixed point by a model string.

For low, high and intermediate speeds, how is the motion of this new pendulum likely to differ from that of the original pendulum whose bob is attached to a fixed point by a light model rod?

The answer to Exercise 19 emphasises that in certain situations equations (41) and (44) apply only while the right-hand side of equation (44) is positive. Depending on the nature of the pendulum, other phenomena might occur if the right-hand side of equation (44) is negative. However, note that a pendulum consisting of a model rod and one consisting of a model string satisfy the same equations, (41) and (44), except that for a model string we require that $T > 0$.

Exercise 20

Consider the motion of a pendulum moving in a vertical plane, whose bob is attached to a fixed point by a model string.

(a) Show that if $v_0 > \sqrt{5gl}$, then the string never goes slack.

(b) If $v_0 = 2\sqrt{gl}$, find the angle θ for which the string goes slack.

In Exercise 20(a) you saw that the string never goes slack if the speed v_0 of the pendulum bob at its lowest point is greater than $\sqrt{5gl}$. In this case, the pendulum bob performs complete revolutions in the same direction. It was also predicted, in Exercise 19, that the string would not go slack and the pendulum would perform oscillatory motion for certain lower speeds. So what is the condition on v_0 for the string to remain taut at lower speeds?

To answer this question, look first at the tension in the string. Recall from equation (43), and the discussion immediately after it, that $T > 0$ whenever the bob is below the pivot. So the string can never go slack before the bob reaches the horizontal.

Above the horizontal, $\cos\theta$ is negative, in which case equation (43) implies that we have $T \leq ml\dot{\theta}^2$. So $T = 0$ before $\dot{\theta} = 0$, that is, the string becomes slack before the bob comes instantaneously to rest.

We know from Exercise 20(a) that the bob reaches the highest point of the circle if $v_0 > \sqrt{5gl}$. We have still to determine the initial speed so that the bob just reaches the horizontal.

Exercise 21

Find an initial speed v_0 so that the pendulum bob just reaches the horizontal when coming to rest.

We have found that the model of a pendulum consisting of a bob attached to a string predicts the following.

- For $0 < v_0 \leq \sqrt{2gl}$, the bob oscillates back and forth with an amplitude of less than or equal to $\pi/2$.
- For $\sqrt{2gl} < v_0 < \sqrt{5gl}$, the bob travels initially in a circular path with the string taut; then at some angle with $\pi/2 < |\theta| < \pi$, the string becomes slack and the bob falls under gravity as a projectile.
- For $v_0 \geq \sqrt{5gl}$, the bob travels continuously in a circle.

Note how this compares to the motion of a pendulum bob attached to a model rod, as described immediately below Exercise 18.

Exercise 22

A smooth hemispherical bowl of radius R, whose lowest point is at A, is fixed with its rim uppermost and horizontal. A particle of mass m is set in motion along the inner surface of the bowl towards A, so its subsequent motion is in a vertical plane through A. The particle is set in motion from a point at a vertical height $\frac{1}{2}R$ above A, with initial speed $\sqrt{gR}$.

(a) Show that the particle will just reach the top of the bowl.

(b) Find the magnitude of the normal reaction of the bowl on the particle when the particle is at a vertical height $\frac{2}{3}R$ above A.

The next two exercises involve noting a particular property of the normal reaction **N**. Recall that when an object lies on a surface, the surface exerts a force **N** on the object, with **N** directed along the outward normal to the surface. You can think of **N** as the reaction of the surface to the object pressing down on it. Clearly, **N** cannot point in the opposite direction, towards the surface, and if the normal reaction vanishes, that is, $\mathbf{N} = \mathbf{0}$, then the object is no longer pressing down on the surface and is on the point of leaving the surface.

Exercise 23

Consider a particle P of mass m, sliding on the outside of a fixed smooth sphere of radius R. Assume that the particle has started from the highest point of the sphere with an initial speed v_0, and that the radius from the centre of the sphere to the particle makes an angle θ with the upward vertical, as shown in Figure 20. Assume also that the particle's path lies in a vertical plane.

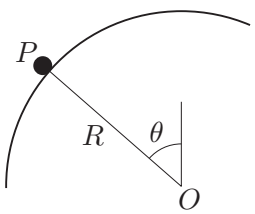


Figure 20 Particle sliding on the outside of a sphere

If $v_0 = \sqrt{\frac{1}{2}gR}$, show that the particle leaves the surface of the sphere when $\theta = \arccos\frac{5}{6} \simeq 33.6°$.

(*Hint*: When considering the motion of the particle in a vertical plane from an equilibrium position at the highest point on the sphere, measuring θ anticlockwise from the upward vertical is similar to measuring θ anticlockwise from the downward vertical in the case of a pendulum.)

Exercise 24

The axis of a smooth fixed cylinder of radius R is horizontal. Two particles P_1, of mass $3m$, and P_2, of mass $5m$, are joined by a model string of length $\pi R/2$ and placed on the surface of the cylinder in a vertical plane perpendicular to the axis of the cylinder, as shown in Figure 21. Initially, the position of the particles is symmetrical about the vertical plane containing the axis of the cylinder and the origin O, with both OP_1 and OP_2 inclined at an angle of $\pi/4$ to the upward vertical through O. The particles are now released from rest, and θ is the angle between OP_1 and the horizontal t seconds after release.

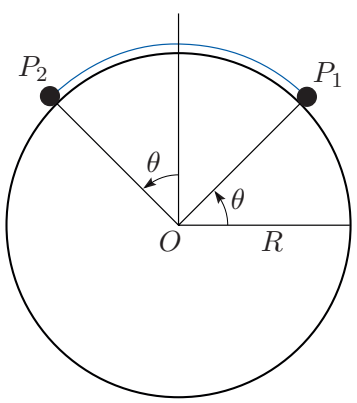


Figure 21 Particles P_1 and P_2 joined by a string and placed on the surface of a cylinder with axis passing through O. The points P_1, P_2 and O lie in the same vertical plane.

For parts (a) and (b), assume that both particles are on the surface of the cylinder, and note that the angle that OP_2 makes with the vertical at time t is also θ.

(a) Show that $|\mathbf{N}|$, the magnitude of the normal reaction of the cylinder on P_2, and $\dot{\theta}$ are related by

$$|\mathbf{N}| = 5m(g\cos\theta - R\dot{\theta}^2).$$

(*Hint*: Consider the forces acting on particle P_2 at time t, and the acceleration of P_2.)

(b) By assuming that the total mechanical energy of the combined system is conserved, show that the speed v of the particles satisfies

$$v^2 = \tfrac{1}{4}gR(4\sqrt{2} - 5\cos\theta - 3\sin\theta).$$

(c) Hence show that the angle θ at which P_2 leaves the cylinder is the solution of

$$9\cos\theta + 3\sin\theta = 4\sqrt{2} \quad (0 < 0 < \pi/2).$$

By re-expressing the left-hand side of this equation in terms of a single trigonometric function, show that this implies that

$$\theta = \arccos\left(\frac{4}{3\sqrt{5}}\right) + \arctan\left(\frac{1}{3}\right) \simeq 1.254 \ (\simeq 71.8^\circ).$$

4 Angular velocity and angular momentum

This section starts to look beyond circular motion to more general rotational motion. In this section we define two key quantities that will be useful in Unit 21, which looks at the motion of rigid bodies.

Subsection 4.1 defines the *angular velocity* of a particle. The magnitude of the angular velocity of a particle is the angular speed that we have been considering up to this point.

Linear momentum is defined both for a single particle and for an n-particle system in Unit 19.

In Unit 19 you were introduced to linear momentum and its conservation in the absence of external forces. You saw how the conservation of linear momentum could be used in the analysis of certain problems involving linear motion. For circular motion, and for rotational motion in general, the corresponding concept is *angular momentum*. Subsection 4.2 defines angular momentum for a particle and shows how it is linked to torques.

4.1 Angular velocity

The rotation of rigid bodies is considered in Unit 21.

If a particle is moving in a circle or if a rigid body is rotating about a fixed axis, then the (angular) rate of rotation and the direction of the axis of rotation are of primary importance. These two quantities can be encapsulated in the vector $\boldsymbol{\omega}$, known as the *angular velocity*. The magnitude of $\boldsymbol{\omega}$ is the angular speed, and the direction of $\boldsymbol{\omega}$ is defined to be along the axis of rotation in the sense given by the right-hand grip rule (or right-hand thumbs-up rule), that is, in the direction in which the thumb of a right hand would point if the fingers were curled in the direction of rotation (see Figure 22).

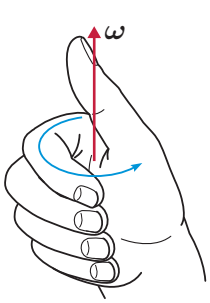


Figure 22 Right-hand grip rule

Angular velocity

The **angular velocity** $\boldsymbol{\omega}$ of a particle moving in a circle, or of a rigid body rotating about a fixed axis, is a vector with magnitude equal to the angular speed and with direction along the axis of rotation in the sense given by the right-hand grip rule.

Example 3

The Earth rotates in an easterly direction once in a sidereal day (the rotation period with respect to the fixed stars, which is less than that with respect to the Sun by one part in 365.25), which is 23.9343 hours.

Specify the angular velocity $\boldsymbol{\omega}$ of the rotation of the Earth.

Solution

The angular speed (in $\mathrm{rad\,s^{-1}}$) is

$$\omega = \frac{2\pi}{23.9343 \times 60 \times 60} \simeq 7.292 \times 10^{-5}.$$

The direction of the angular velocity $\boldsymbol{\omega}$ is from the South Pole to the North Pole, as shown in Figure 23.

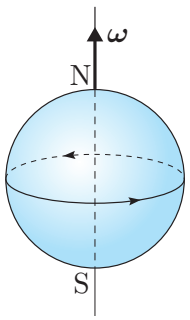


Figure 23 Rotation of the Earth

The direction of the angular velocity $\boldsymbol{\omega}$ is along the axis of rotation and is therefore perpendicular to the plane of motion. To be more specific about this direction, we use the usual right-handed system shown in Figure 24. Using the right-hand grip rule, the direction of the angular velocity vector $\boldsymbol{\omega}$ will be in the positive $\mathbf{k}$-direction if $\dot{\theta} > 0$ and in the negative $\mathbf{k}$-direction if $\dot{\theta} < 0$, as shown in Figure 25(a) and Figure 25(b), respectively.

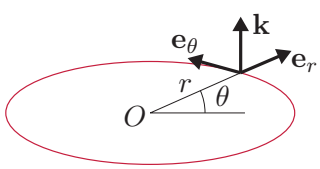
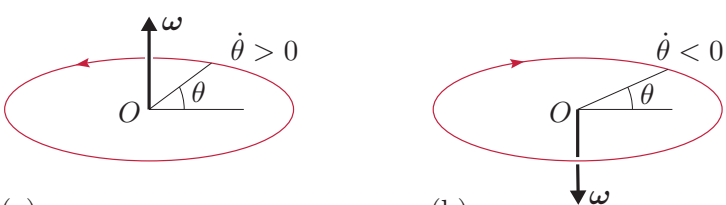


Figure 24 Right-handed rotating system



(a) (b)

Figure 25 Rotating system with (a) $\dot{\theta} > 0$ and (b) $\dot{\theta} < 0$

Therefore

$$\boldsymbol{\omega} = \begin{cases} \omega\mathbf{k} & (\dot{\theta} > 0) \\ -\omega\mathbf{k} & (\dot{\theta} < 0) \end{cases} = \begin{cases} |\dot{\theta}|\mathbf{k} & (\dot{\theta} > 0) \\ -|\dot{\theta}|\mathbf{k} & (\dot{\theta} < 0) \end{cases} = \dot{\theta}\mathbf{k}.$$

So the relationship $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$ holds, regardless of the direction of rotation.

The relationship also holds if there is no rotation, when $\dot{\theta} = 0$ and $\boldsymbol{\omega} = \mathbf{0}$.

Relationship between angular velocity and the axis and rate of rotation

The angular velocity $\boldsymbol{\omega}$ of a particle moving in a circle about an axis in the $\mathbf{k}$-direction, with angular rate of rotation given by $\dot{\theta}$, is

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{k}. \tag{45}$$

We complete this subsection by establishing the link between the angular velocity $\boldsymbol{\omega}$ of a particle in circular motion and its velocity $\mathbf{v} = \dot{\mathbf{r}}$. For circular motion of radius R, we know from equation (19) that $\mathbf{r} = R\mathbf{e}_r$, relative to the centre of the circle, and from equation (20) that $\mathbf{v} = \dot{\mathbf{r}} = R\dot{\theta}\mathbf{e}_\theta$. We also know that $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$. Therefore

You can check that $\mathbf{k} \times \mathbf{e}_r = \mathbf{e}_\theta$ by looking at Figure 24.

$$\boldsymbol{\omega} \times \mathbf{r} = \dot{\theta}\mathbf{k} \times R\mathbf{e}_r = R\dot{\theta}(\mathbf{k} \times \mathbf{e}_r) = R\dot{\theta}\mathbf{e}_\theta = \mathbf{v}.$$

This result also holds if we take the origin to be anywhere on the axis of rotation. For then $\mathbf{r} = R\mathbf{e}_r + z\mathbf{k}$ for some z, and

Recall from Unit 2 that $\mathbf{k} \times \mathbf{k} = \mathbf{0}$.

$$\begin{aligned}\boldsymbol{\omega} \times \mathbf{r} &= (\dot{\theta}\mathbf{k}) \times (R\mathbf{e}_r + z\mathbf{k}) \\ &= \dot{\theta}\mathbf{k} \times R\mathbf{e}_r + \dot{\theta}\mathbf{k} \times z\mathbf{k} \\ &= R\dot{\theta}(\mathbf{k} \times \mathbf{e}_r) \\ &= R\dot{\theta}\mathbf{e}_\theta \\ &= \mathbf{v}.\end{aligned}$$

Relationship between velocity and angular velocity

The velocity $\mathbf{v}$ of a particle moving in a circle with angular velocity $\boldsymbol{\omega}$ is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \tag{46}$$

where $\mathbf{r}$ is the position vector of the particle relative to an origin on the axis of rotation.

Exercise 25

A particle moves clockwise in a circle at a constant angular speed of $2\,\text{rad}\,\text{s}^{-1}$.

(a) Find the angular velocity of the particle.

(b) If the position vector of the particle, relative to the centre of the circle, at a certain instant is $3\mathbf{i} + 4\mathbf{j}$, find its velocity at that instant both in terms of the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$, and in terms of the corresponding polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

4.2 Angular momentum

Consider a particle of mass m moving along a curve in space. This is illustrated in Figure 26.

At time t the particle has position vector $\mathbf{r} = \mathbf{r}(t)$, velocity $\dot{\mathbf{r}} = \dot{\mathbf{r}}(t)$ and acceleration $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}(t)$ relative to a fixed Cartesian coordinate system. The linear momentum of the particle is

$$\mathbf{p} = m\dot{\mathbf{r}}.$$

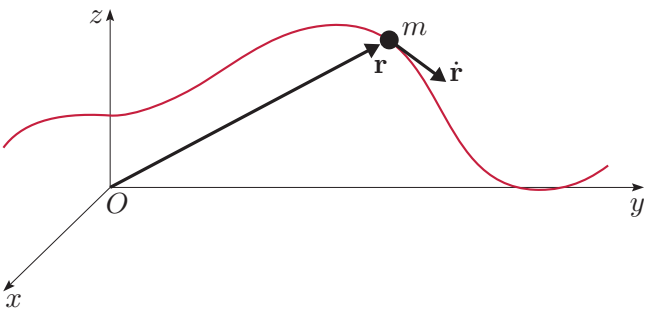


Figure 26 Particle with position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$, moving along a curve

You saw in Unit 19 that the linear momentum is related to the force $\mathbf{F}$ acting on the particle according to the equation

$$\mathbf{F} = \dot{\mathbf{p}}, \tag{47}$$

This equation is Newton's second law $\mathbf{F} = m\ddot{\mathbf{r}}$ expressed in terms of linear momentum.

which says that the rate of change of linear momentum of the particle is equal to the force acting on the particle.

In Unit 2 we quantified the turning effect of a force about a fixed point O by defining the torque $\boldsymbol{\Gamma}$ of the force about O as

$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}.$$

By analogy with the case of linear momentum, when analysing rotational motion it would be useful to have a quantity that plays the same role in relation to torques as linear momentum does in relation to forces. So by analogy with equation (47), we would like the derivative of this quantity to equal the torque on the particle. A suitable quantity is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\dot{\mathbf{r}},$$

since, using the rule for differentiating a cross product, we have

$$\begin{aligned}\dot{\mathbf{L}} &= \frac{d}{dt}(\mathbf{r} \times m\dot{\mathbf{r}}) \\ &= (\dot{\mathbf{r}} \times m\dot{\mathbf{r}}) + (\mathbf{r} \times m\ddot{\mathbf{r}}) \\ &= \mathbf{r} \times m\ddot{\mathbf{r}} \\ &= \mathbf{r} \times \mathbf{F} \\ &= \boldsymbol{\Gamma}.\end{aligned}$$

The cross product of parallel vectors is zero, so $\dot{\mathbf{r}} \times m\dot{\mathbf{r}} = \mathbf{0}$.

We call $\mathbf{L} = \mathbf{L}(t)$ the *angular momentum* of the particle about the fixed point.

Angular momentum

For a particle that has linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ and position vector $\mathbf{r}$ relative to an origin O, its **angular momentum** $\mathbf{L}$ about O is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\dot{\mathbf{r}}. \tag{48}$$

The SI units for the magnitude of angular momentum are $\mathrm{kg\,m^2\,s^{-1}}$.

As with torques, the angular momentum of a particle is dependent on the choice of origin O. Figure 27 illustrates this, where the vector $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$ represents the angular momentum relative to the origin O, while the vector $\mathbf{L}' = \mathbf{r}' \times m\dot{\mathbf{r}}'$ represents the angular momentum relative to the origin O'. These vectors may differ in both magnitude and direction. (However, since both coordinate systems are static, the velocity of the particle is the same in each case, i.e. $\dot{\mathbf{r}} = \dot{\mathbf{r}}'$.)

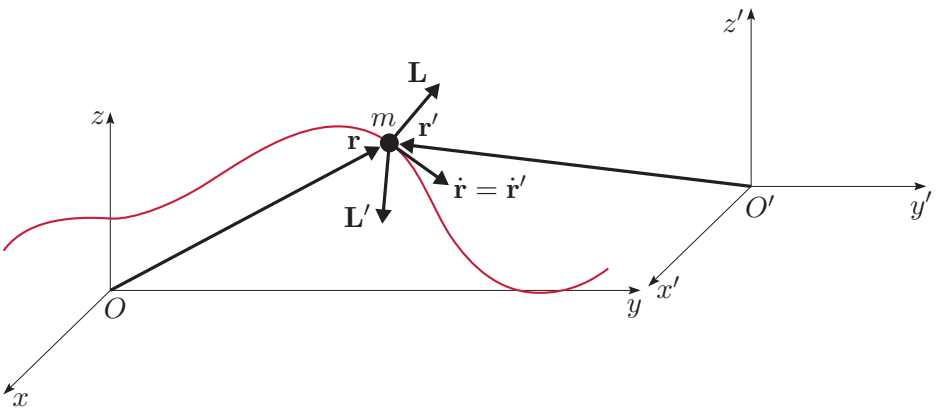


Figure 27 Angular momentum relative to different static origins O and O'

The fact that $\dot{\mathbf{L}} = \mathbf{\Gamma}$ is given a special name.

The torque law is fundamental to the study of rotating bodies.

Torque law for a particle

The rate of change of a particle's angular momentum about a fixed point is equal to the applied torque about that point, that is,

$$\dot{\mathbf{L}} = \mathbf{\Gamma}.$$

Example 4

A particle of mass m, on which no forces act, moves with constant speed u parallel to the y-axis in the positive y-direction in the (x, y)-plane. At time $t = t_0$, its path intersects the x-axis at $x = b$ (see Figure 28).

Calculate the magnitude and direction of the angular momentum of the particle with respect to the origin O.

Solution

The position of the particle in the coordinate system of Figure 28 is

$$\mathbf{r} = b\mathbf{i} + u(t - t_0)\mathbf{j},$$

while the velocity is $\dot{\mathbf{r}} = u\mathbf{j}$.

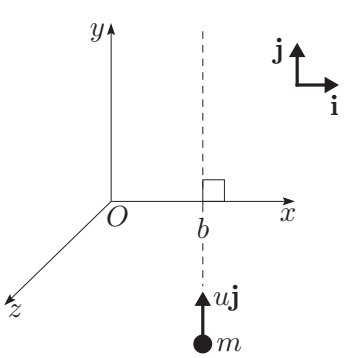


Figure 28 Particle moving parallel to the y-axis at constant speed

So using $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, the angular momentum is

$$\begin{aligned}\mathbf{L} &= \mathbf{r} \times m\dot{\mathbf{r}} \\ &= (b\mathbf{i} + u(t - t_0)\mathbf{j}) \times mu\mathbf{j} \\ &= bmu\mathbf{k}.\end{aligned}$$

Thus even though the vector $\mathbf{r}$ is changing continuously, the cross product $\mathbf{r} \times m\dot{\mathbf{r}}$ is a constant vector.

Hence the angular momentum points along the positive z-axis and has magnitude bmu.

Exercise 26

At a given time, a particle of mass 3 has position $\mathbf{i} + 2\mathbf{j}$ and velocity $2\mathbf{i} - \mathbf{j}$.

What is the particle's angular momentum about the origin of the coordinate system at this time?

All quantities are measured in SI units.

Example 5

A particle of mass m at position $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ moves in a circle, whose centre lies on the z-axis, with angular velocity $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$.

Find the angular momentum $\mathbf{L}$ of the particle about the origin O in terms of x, y, z, $\dot{\theta}$, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Solution

The motion of the particle is shown in Figure 29. By definition, we have $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$. From equation (46), the velocity is

$$\begin{aligned}\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r} &= \dot{\theta}\mathbf{k} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= \dot{\theta}(x\mathbf{j} - y\mathbf{i}).\end{aligned}$$

It follows that

$$\begin{aligned}\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}} &= (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times m\dot{\theta}(x\mathbf{j} - y\mathbf{i}) \\ &= m\dot{\theta}(-xz\mathbf{i} - yz\mathbf{j} + (x^2 + y^2)\mathbf{k}).\end{aligned}$$

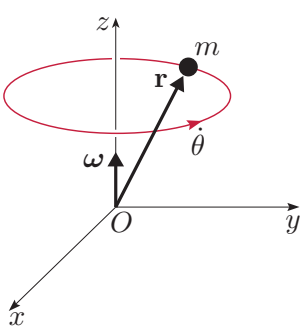


Figure 29 Particle moving in a circle

Example 5 demonstrates that even for circular motion, the direction of the angular momentum $\mathbf{L}$ is *not*, in general, the same as the direction of the angular velocity $\boldsymbol{\omega}$. However, if the origin is at the centre of the circular motion, then the angular velocity and the angular momentum of the particle are in the same direction, since then $z = 0$ in Example 5 and $\mathbf{L} = m\dot{\theta}(x^2 + y^2)\mathbf{k} = mR^2\dot{\theta}\mathbf{k}$, where R is the radius of the circular motion. Note that we can write $\mathbf{L} = mR^2\boldsymbol{\omega}$.

These results hold even when $\boldsymbol{\omega}$ is not constant.

The relationship between angular velocity and angular momentum will be explored further in Unit 21.

This result can be extended to general motion confined to the (x, y)-plane, not just circular motion. If the particle has position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = r\mathbf{e}_r,$$

where r, the distance to the origin, is not necessarily constant, then using the results of Subsection 1.2 we have for the velocity

$$\dot{\mathbf{r}} = \frac{d}{dt}(r\mathbf{e}_r) = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta,$$

where we have used $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$ from equation (16). Hence the angular momentum about the origin is

$$\begin{aligned}\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}} &= mr\mathbf{e}_r \times (\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta) \\ &= mr(\dot{r}\mathbf{e}_r \times \mathbf{e}_r + r\dot{\theta}\mathbf{e}_r \times \mathbf{e}_\theta) \\ &= mr^2\dot{\theta}\mathbf{k}.\end{aligned}$$

We have used $\mathbf{e}_r \times \mathbf{e}_r = \mathbf{0}$ and $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{k}$.

Angular momentum for general planar motion

The angular momentum $\mathbf{L}$ about the origin of a particle of mass m with motion (not necessarily circular) confined to the (x, y)-plane is

$$\mathbf{L} = mr^2\dot{\theta}\mathbf{k}. \tag{49}$$

Exercise 27

A particle of mass m moves in the (x, y)-plane in a circle of radius R, whose centre is at the origin, with angular velocity $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$.

Show that $|\mathbf{L}| = m|\mathbf{r}|\,|\dot{\mathbf{r}}|$, where $\mathbf{L}$ is the angular momentum of the particle about the origin, $\mathbf{r}$ is its position relative to the origin, and $\dot{\mathbf{r}}$ is its velocity.

Example 6

A pendulum bob of mass m is fixed to an origin O by a taut light string of constant length l. The bob swings to and fro in a vertical (x, y)-plane.

Use the torque law to derive the equation of motion of the bob.

Solution

Model the bob as a particle and the string as a model string. Use a polar coordinate system with origin O at the centre of the motion, and let $\mathbf{k}$ be a unit vector pointing out of the page for the set-up shown in Figure 30.

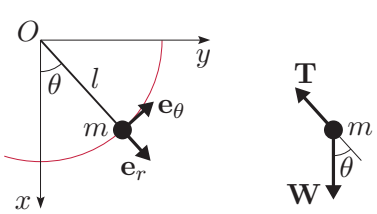


Figure 30 Pendulum swinging in a vertical plane

The total force $\mathbf{F}$ acting on the particle is the sum of its weight $\mathbf{W}$ and the tension force $\mathbf{T}$ due to the string, that is,

$$\mathbf{F} = \mathbf{W} + \mathbf{T},$$

where $\mathbf{W} = (mg\cos\theta)\mathbf{e}_r - (mg\sin\theta)\mathbf{e}_\theta$ and $\mathbf{T} = -|\mathbf{T}|\mathbf{e}_r$. The corresponding torque about O is

$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{W} + \mathbf{T}) = l\mathbf{e}_r \times ((mg\cos\theta)\mathbf{e}_r - (mg\sin\theta)\mathbf{e}_\theta - |\mathbf{T}|\mathbf{e}_r)$$
$$= -(mgl\sin\theta)\mathbf{k}.$$

From equation (49), with $r = l$, the angular momentum of the particle about O is

$$\mathbf{L} = ml^2\dot{\theta}\mathbf{k}.$$

Hence, by the torque law, we have

$$\frac{d}{dt}(ml^2\dot{\theta}\mathbf{k}) = -(mgl\sin\theta)\mathbf{k}.$$

Differentiating, resolving in the $\mathbf{k}$-direction and rearranging, we obtain the equation of motion

$$\ddot{\theta} = -\frac{g}{l}\sin\theta.$$

This result is the same as that obtained by different means in Unit 11 and in Section 3.

Exercise 28

The position of a particle of mass m at time t is given by

$$\mathbf{r}(t) = a\cos(\alpha t)\mathbf{i} + a\sin(\alpha t)\mathbf{j} + ut\mathbf{k},$$

where a, α and u are constants. The particle is moving in a spiral path as shown in Figure 31.

Write down expressions for the linear momentum of the particle, the force acting on the particle, the angular momentum of the particle about the origin O, and the torque about O acting on the particle. Verify the torque law for the motion of the particle.

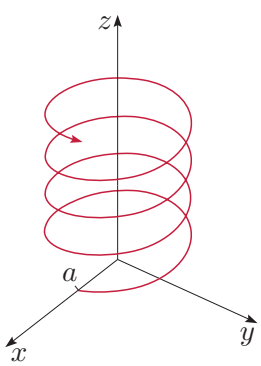


Figure 31 Particle moving in a spiral path

Exercise 29

In the film *The Wizard of Oz*, a house is swept up in a whirlwind. After a while, the whirlwind begins to dissipate and the house is subject to a decreasing torque. Although the house moves slowly towards the axis of the whirlwind, its motion is almost circular around this axis.

Model the house as a particle of mass m performing anticlockwise circular motion of radius R around a vertical z-axis in a horizontal (x, y)-plane, as shown in Figure 32. Model the decreasing torque by $\boldsymbol{\Gamma} = \Gamma_0 e^{-\alpha t}\mathbf{k}$, where Γ_0 and α are positive constants, t represents time, and $\mathbf{k}$ is a unit vector in the positive z-direction.

(a) Find an expression for the angular speed of the particle in terms of the other variables and parameters. To what value does this angular speed tend as t increases?

(b) Suppose that we no longer ignore the motion towards the axis of the whirlwind, and allow R to become smaller and smaller. How does this affect the angular speed? As R becomes smaller, does it remain valid to model the house as a particle?

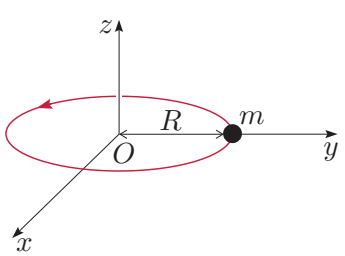


Figure 32 House swept up by a whirlwind

The most important special case for the torque law is the case when no resultant torque acts on a particle. In this case the torque law $\dot{\mathbf{L}} = \boldsymbol{\Gamma}$ reduces to $\dot{\mathbf{L}} = \mathbf{0}$, so $\mathbf{L}$ is a constant vector. So if there is no resultant torque, then the angular momentum is constant. This is usually stated in the following form.

Law of conservation of angular momentum for a particle

If the total torque acting on a particle about a fixed point is zero, then the angular momentum of the particle about that point is constant. In other words, if $\boldsymbol{\Gamma} = \mathbf{0}$, then $\mathbf{L}$ is constant.

The total torque acting on a particle will be zero if the total force acting on the particle is zero, but this is not an especially interesting situation. More noteworthy is the case when the total force acting on a particle is directed towards a fixed point, which we can choose to be the origin. A force of this type is called a **central force**, and in the case where it is also spherically symmetric, it can be expressed as $\mathbf{F} = f(r)\,\widehat{\mathbf{r}}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = |\mathbf{r}|$ and $\widehat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. The gravitational force given in equation (26) is an important example of a central force.

Central force fields were introduced in Unit 16 and were shown to be conservative.

Exercise 30

Show that the total torque of the central force $\mathbf{F} = f(r)\,\widehat{\mathbf{r}}$ about the origin is zero.

Exercise 31

Consider a particle moving around a circle at constant speed.

Show that the total torque about the centre of the circle is zero.

Learning outcomes

After studying this unit, you should be able to:

- differentiate products of vector functions
- understand and use the polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$
- understand and use the expressions for position, velocity and acceleration for the circular motion of a particle in polar coordinates
- understand and use Newton's law of universal gravitation
- understand and use angular velocity in the description of circular motion
- relate the angular velocity to the velocity of a particle in circular motion
- use polar unit vectors to solve mechanics problems involving uniform and non-uniform circular motion
- understand and use the concept of angular momentum for the motion of a particle
- use the torque law to solve simple mechanics problems involving a single particle.

Solutions to exercises

Solution to Exercise 1

(a) Differentiating and noting that $\mathbf{i}$ and $\mathbf{j}$ are constant vectors, we find that

$$\dot{\mathbf{r}}(t) = -12\sin(4t)\mathbf{i} + 12\cos(4t)\mathbf{j},$$
$$\ddot{\mathbf{r}}(t) = -48\cos(4t)\mathbf{i} - 48\sin(4t)\mathbf{j}.$$

(b) The magnitude of the acceleration vector is

$$|\ddot{\mathbf{r}}(t)| = \sqrt{(-48\cos(4t))^2 + (-48\sin(4t))^2}$$
$$= \sqrt{48^2(\cos^2(4t) + \sin^2(4t))} = 48.$$

(c) The acceleration vector can be written as

$$-16(3\cos(4t)\mathbf{i} + 3\sin(4t)\mathbf{j}) = -16\mathbf{r}(t).$$

So $\ddot{\mathbf{r}}(t) = -c\,\mathbf{r}(t)$, with $c = 16$.

Solution to Exercise 2

The magnitude of the sideways force, towards the centre of the circle, exerted on the locomotive by the track is $|\mathbf{F}| = m|\mathbf{a}| = m\omega^2 R$. We have $m = 40$ and $R = 100$. If one circuit is completed in 50 s, then the angular speed is $\omega = 2\pi/50$. So the magnitude of the force is

$$40\left(\frac{2\pi}{50}\right)^2 100 \simeq 63.17.$$

So the magnitude of the sideways force exerted by the track on the locomotive is 63.17 N.

Solution to Exercise 3

We write $\mathbf{f}$ and $\mathbf{g}$ in component form as

$$\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}, \quad \mathbf{g} = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k},$$

so

$$\mathbf{f} \cdot \mathbf{g} = f_1g_1 + f_2g_2 + f_3g_3.$$

Using the formula for differentiating the product of two scalar functions, we obtain

$$\begin{aligned}\frac{d}{dt}(\mathbf{f}\cdot\mathbf{g}) &= \frac{d}{dt}(f_1g_1)+\frac{d}{dt}(f_2g_2)+\frac{d}{dt}(f_3g_3)\\ &= \left(\frac{df_1}{dt}\,g_1+f_1\,\frac{dg_1}{dt}\right)+\left(\frac{df_2}{dt}\,g_2+f_2\,\frac{dg_2}{dt}\right)+\left(\frac{df_3}{dt}\,g_3+f_3\,\frac{dg_3}{dt}\right)\\ &= \left(\frac{df_1}{dt}\,g_1+\frac{df_2}{dt}\,g_2+\frac{df_3}{dt}\,g_3\right)+\left(f_1\,\frac{dg_1}{dt}+f_2\,\frac{dg_2}{dt}+f_3\,\frac{dg_3}{dt}\right)\\ &= \left(\frac{df_1}{dt}\mathbf{i}+\frac{df_2}{dt}\mathbf{j}+\frac{df_3}{dt}\mathbf{k}\right)\cdot(g_1\mathbf{i}+g_2\mathbf{j}+g_3\mathbf{k})\\ &\quad+(f_1\mathbf{i}+f_2\mathbf{j}+f_3\mathbf{k})\cdot\left(\frac{dg_1}{dt}\mathbf{i}+\frac{dg_2}{dt}\mathbf{j}+\frac{dg_3}{dt}\mathbf{k}\right)\\ &= \frac{d\mathbf{f}}{dt}\cdot\mathbf{g}+\mathbf{f}\cdot\frac{d\mathbf{g}}{dt},\end{aligned}$$

as required.

Solution to Exercise 4

Using equation (7),

$$\frac{d}{dt}(\mathbf{f}\cdot\mathbf{f})=\frac{d\mathbf{f}}{dt}\cdot\mathbf{f}+\mathbf{f}\cdot\frac{d\mathbf{f}}{dt}=2\mathbf{f}\cdot\frac{d\mathbf{f}}{dt}.$$

Solution to Exercise 5

Differentiating equation (13) with respect to time using the chain rule, we have

$$\dot{\mathbf{e}}_\theta=(-\cos\theta)\dot\theta\mathbf{i}+(-\sin\theta)\dot\theta\mathbf{j}=-\dot\theta((\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j})=-\dot\theta\mathbf{e}_r,$$

using equation (12).

Solution to Exercise 6

Differentiating equation (18) with respect to time (where θ and $\mathbf{e}_\theta$ are functions of time) using the product rule, we obtain

$$\ddot{\mathbf{r}}=R\ddot\theta\mathbf{e}_\theta+R\dot\theta\dot{\mathbf{e}}_\theta,$$

which, by equation (17), may be written as

$$\ddot{\mathbf{r}}=R\ddot\theta\mathbf{e}_\theta+(R\dot\theta)(-\dot\theta\mathbf{e}_r)=-R\dot\theta^2\mathbf{e}_r+R\ddot\theta\mathbf{e}_\theta.$$

Solution to Exercise 7

The radius of the circle is $R=1$, and $\theta(t)=t^2$. Hence $\dot\theta=2t$ and $\ddot\theta=2$. Therefore the velocity of the particle is (from equation (20))

$$\dot{\mathbf{r}}=R\dot\theta\mathbf{e}_\theta=2t\mathbf{e}_\theta,$$

and the acceleration is (from equation (21))

$$\ddot{\mathbf{r}}=-R\dot\theta^2\mathbf{e}_r+R\ddot\theta\mathbf{e}_\theta=-4t^2\mathbf{e}_r+2\mathbf{e}_\theta.$$

Solution to Exercise 8

The minute hand of a clock makes one complete revolution (clockwise when viewed from the front of the clock) in 1 hour = 3600 seconds. So the fly's angular speed (in $\mathrm{rad\,s^{-1}}$) is

$$\omega = \frac{2\pi}{3600} \simeq 1.745 \times 10^{-3}.$$

The fly's speed (in $\mathrm{m\,s^{-1}}$) is

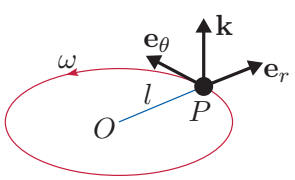
$$v = R\omega = (0.5)\frac{2\pi}{3600} \simeq 8.727 \times 10^{-4}.$$

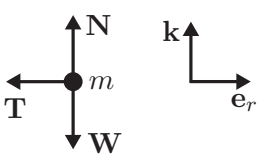
The acceleration has magnitude (in $\mathrm{m\,s^{-2}}$)

$$R\omega^2 = 0.5\left(\frac{2\pi}{3600}\right)^2 \simeq 1.523 \times 10^{-6}.$$

Solution to Exercise 9

Use a polar coordinate system with origin O at the fixed end of the string, and let $\mathbf{k}$ be a unit vector pointing vertically upwards, as shown in the figure below.





There are three forces acting on the particle: the tension force $\mathbf{T} = -|\mathbf{T}|\mathbf{e}_r$ due to the string, the normal reaction $\mathbf{N} = |\mathbf{N}|\mathbf{k}$ of the table on the particle, and its weight $\mathbf{W} = -mg\mathbf{k}$. As the particle is in uniform circular motion with radius l and angular speed ω, the acceleration is, using equation (23),

$$\ddot{\mathbf{r}} = -l\omega^2\mathbf{e}_r.$$

Applying Newton's second law gives

$$m\ddot{\mathbf{r}} = \mathbf{T} + \mathbf{N} + \mathbf{W},$$

so we have

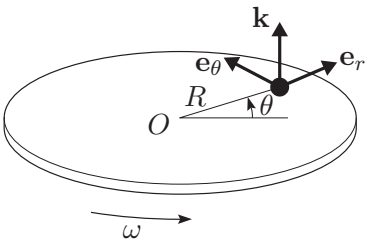
$$-ml\omega^2\mathbf{e}_r = -|\mathbf{T}|\mathbf{e}_r + |\mathbf{N}|\mathbf{k} - mg\mathbf{k}.$$

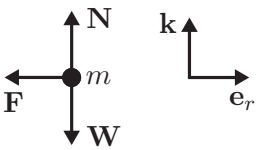
Resolving in the $\mathbf{e}_r$-direction gives the required result,

$$|\mathbf{T}| = ml\omega^2.$$

Solution to Exercise 10

Use a polar coordinate system in the plane of the surface of the disc, with origin O at the centre of the disc, and let $\mathbf{k}$ be a unit vector pointing vertically upwards, as shown in the figure below.





If the disc rotates through one complete revolution every two seconds, then its angular speed is

$$\omega = 2\pi/2 = \pi.$$

The forces acting on the particle are its weight $\mathbf{W} = -mg\mathbf{k}$, the normal reaction $\mathbf{N} = |\mathbf{N}|\mathbf{k}$ from the disc, and the friction force $\mathbf{F}$. The direction of $\mathbf{F}$ is in the horizontal plane of the disc.

Since the particle stays at the same point on the disc, at a distance R, say, from the centre, it performs uniform circular motion. Hence its acceleration is, using equation (23),

$$\ddot{\mathbf{r}} = -R\omega^2\mathbf{e}_r.$$

Applying Newton's second law to the particle, we obtain

$$m\ddot{\mathbf{r}} = \mathbf{W} + \mathbf{N} + \mathbf{F},$$

so we have

$$-mR\omega^2\mathbf{e}_r = -mg\mathbf{k} + |\mathbf{N}|\mathbf{k} + \mathbf{F}.$$

Since the friction force is in the horizontal plane, resolving in the $\mathbf{k}$-direction gives

$$|\mathbf{N}| = mg.$$

Hence we can deduce that

$$\mathbf{F} = -mR\omega^2\mathbf{e}_r,$$

so the friction force is directed towards the centre of the disc and has magnitude

$$|\mathbf{F}| = mR\omega^2.$$

If the particle is to remain static (relative to the disc), we must have

$$|\mathbf{F}| \leq \mu|\mathbf{N}|,$$

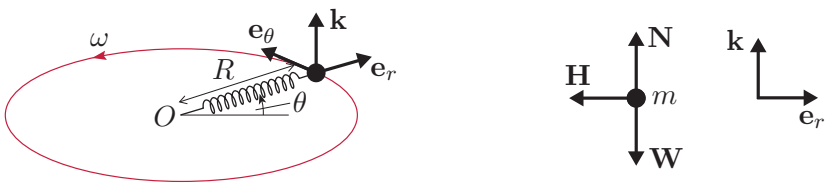
which, from the expressions above for $|\mathbf{F}|$ and $|\mathbf{N}|$, and from $\omega = \pi$ and $\mu = \frac{1}{2}$, leads to

$$R \leq \frac{\mu g}{\omega^2} = \frac{g}{2\pi^2} \simeq 0.497.$$

Thus the maximum possible distance of the toy from the centre of the roundabout is about 0.5 m.

Solution to Exercise 11

Use a polar coordinate system in the horizontal plane of the surface of the table, with origin at O, and take $\mathbf{k}$ to be a unit vector pointing vertically upwards. The forces on the particle are its weight $\mathbf{W}$, the normal reaction $\mathbf{N}$ of the table, and the spring force $\mathbf{H}$. Let R be the length of the spring. The situation is shown in the figure below.



Applying Newton's second law to the particle, we have

$$m\ddot{\mathbf{r}} = \mathbf{W} + \mathbf{N} + \mathbf{H},$$

where $\ddot{\mathbf{r}} = -R\omega^2\mathbf{e}_r$ (using equation (23)), $\mathbf{W} = -mg\mathbf{k}$, $\mathbf{N} = |\mathbf{N}|\mathbf{k}$ and $\mathbf{H} = k(R - l_0)(-\mathbf{e}_r)$.

Resolving in the $\mathbf{e}_r$-direction gives

$$-mR\omega^2 = -k(R - l_0),$$

so the length of the model spring is

$$R = \frac{kl_0}{k - m\omega^2}.$$

(Since $R > 0$, we must have $k - m\omega^2 > 0$, that is, $\omega < \sqrt{k/m}$.)

Solution to Exercise 12

Applying Newton's second law and equation (23) to the planet, we obtain

$$-mR\omega^2\mathbf{e}_r = -\frac{GmM}{R^2}\mathbf{e}_r.$$

Hence

$$\omega = \sqrt{\frac{GM}{R^3}}.$$

So the period of the planet in its orbit is

$$T = \frac{2\pi}{\omega}$$
$$= 2\pi\sqrt{\frac{R^3}{GM}}.$$

It follows that

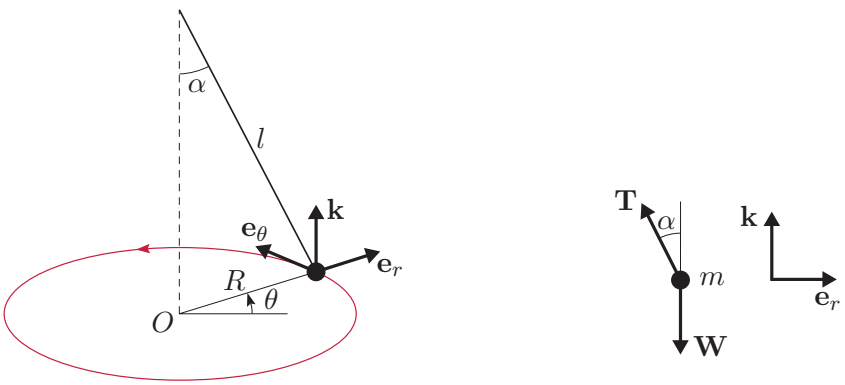
$$T^2 = \frac{4\pi^2}{GM}R^3,$$

that is,

$$T^2 \propto R^3.$$

Solution to Exercise 13

(a) Use a polar coordinate system in the horizontal plane of the particle's motion, with origin O at the centre of its circular path, and take $\mathbf{k}$ to be a unit vector pointing vertically upwards. Let R be the radius of the circular path so $R = l\sin\alpha$. The only forces acting on the particle are its weight $\mathbf{W}$ and the tension force $\mathbf{T}$ due to the rod. The situation is shown in the figure below.



Applying Newton's second law and equation (23) to the particle gives

$$-mR\omega^2\mathbf{e}_r = \mathbf{W} + \mathbf{T},$$

where $\mathbf{T} = -(|\mathbf{T}|\sin\alpha)\mathbf{e}_r + (|\mathbf{T}|\cos\alpha)\mathbf{k}$ and $\mathbf{W} = -mg\mathbf{k}$.

Resolving in the $\mathbf{k}$-direction gives

$$0 = -mg + |\mathbf{T}|\cos\alpha,$$

so the tension in the rod is $|\mathbf{T}| = mg/\cos\alpha$.

(b) Resolving in the $\mathbf{e}_r$-direction gives

$$-mR\omega^2 = -|\mathbf{T}|\sin\alpha.$$

Substituting for $|\mathbf{T}|$ from the result in part (a), dividing through by m and rearranging, we obtain

$$\omega = \sqrt{\frac{g\sin\alpha}{R\cos\alpha}},$$

and using $R = l\sin\alpha$ we finally have

$$\omega = \sqrt{\frac{g}{l\cos\alpha}}.$$

Since ω is real, $\cos\alpha$ must be non-negative, and therefore $0 < \alpha < \pi/2$. Also, the minimal value of ω is determined by the maximal value of $\cos\alpha$, which occurs when $\alpha = 0$, in which case $\cos\alpha = 1$. Hence $\omega > \omega_0 = \sqrt{g/l}$.

Rearranging the equation above gives

$$\alpha = \arccos\left(\frac{g}{l\omega^2}\right),$$

from which it follows that $\alpha \to \arccos(0) = \pi/2$ as $\omega \to \infty$.

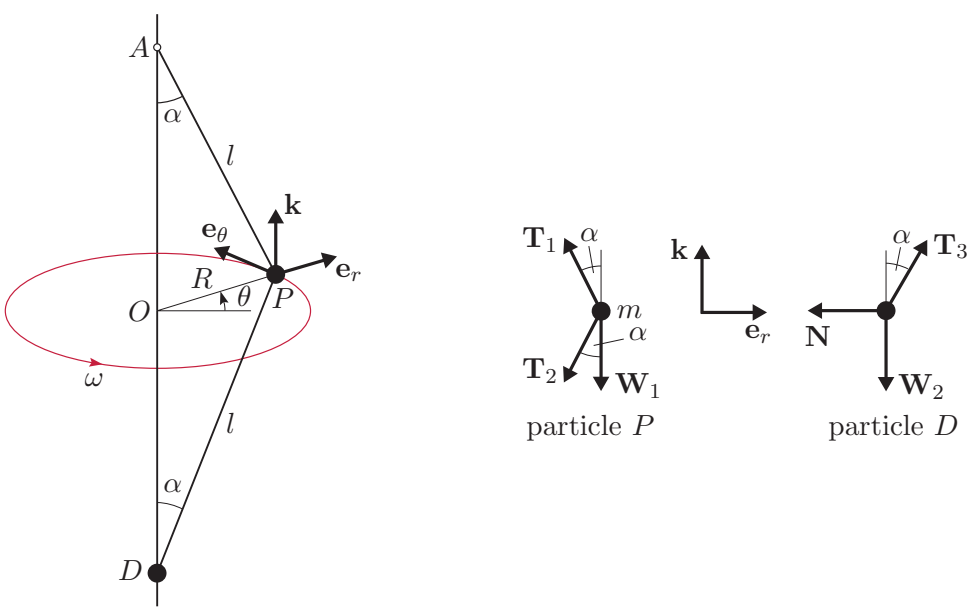
(c) The vertical distance h of the bob below the pivot is given by $h = l\cos\alpha$, but from above we have $\omega = \sqrt{g/h}$, so $h = g/\omega^2$, which is independent of l for a given ω.

See Unit 9.

Note that this means that the potential energy of the conical pendulum, which is given by the height above the datum times mg, does not change as l is varied if ω is kept fixed. Thus modifying l while keeping ω fixed changes the system's kinetic energy but not its potential energy.

Solution to Exercise 14

(a) Use a polar coordinate system in the horizontal plane of the motion of particle P, with origin O at the centre of its circular path, and take $\mathbf{k}$ to be a unit vector pointing vertically upwards. Let R be the radius of the circular path. The forces acting on particle P are its weight $\mathbf{W}_1$, and the tension forces $\mathbf{T}_1$ and $\mathbf{T}_2$ due to the rods PA and PD, respectively. The forces acting on particle D are its weight $\mathbf{W}_2$, the tension force $\mathbf{T}_3$ due to the rod DP, and the normal reaction $\mathbf{N}$ of the shaft on the particle. The situation is shown in the figure below.



Applying Newton's second law and equation (23) to particle P gives

$$-mR\omega^2\mathbf{e}_r = \mathbf{W}_1 + \mathbf{T}_1 + \mathbf{T}_2,$$

where $\mathbf{T}_1 = -(|\mathbf{T}_1|\sin\alpha)\mathbf{e}_r + (|\mathbf{T}_1|\cos\alpha)\mathbf{k}$, $\mathbf{W}_1 = -mg\mathbf{k}$ and $\mathbf{T}_2 = -(|\mathbf{T}_2|\sin\alpha)\mathbf{e}_r - (|\mathbf{T}_2|\cos\alpha)\mathbf{k}$.

For fixed ω and α, particle D is in equilibrium, so

$$\mathbf{W}_2 + \mathbf{T}_3 + \mathbf{N} = \mathbf{0},$$

where $\mathbf{W}_2 = -mg\mathbf{k}$, $\mathbf{T}_3 = -\mathbf{T}_2$ (by Newton's third law) and $\mathbf{N} = -|\mathbf{N}|\mathbf{e}_r$.

We want to find $|\mathbf{T}_1|$ and $|\mathbf{T}_2| = |\mathbf{T}_3|$. Resolving the above equations in the $\mathbf{k}$-direction gives, respectively,

$$0 = -mg + |\mathbf{T}_1|\cos\alpha - |\mathbf{T}_2|\cos\alpha,$$
$$-mg + |\mathbf{T}_2|\cos\alpha = 0$$

(since $\mathbf{N}$ is horizontal, it has no $\mathbf{k}$-component).

Hence $|\mathbf{T}_2| = |\mathbf{T}_3| = mg/\cos\alpha$ and $|\mathbf{T}_1| = 2mg/\cos\alpha$.

(b) Resolving the particle P equation in the $\mathbf{e}_r$-direction gives

$$-mR\omega^2 = -|\mathbf{T}_1|\sin\alpha - |\mathbf{T}_2|\sin\alpha.$$

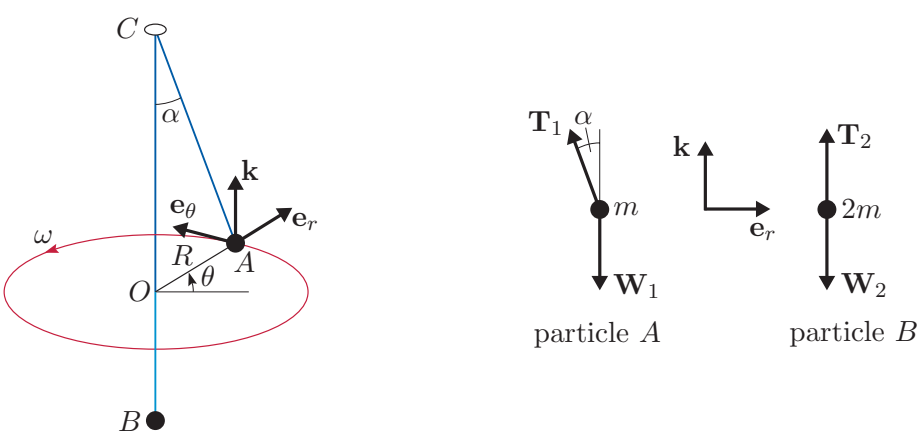
Substituting the values found in part (a) for $|\mathbf{T}_1|$ and $|\mathbf{T}_2|$, and using $R = l\sin\alpha$, we obtain

$$-m(l\sin\alpha)\omega^2 = -\frac{3mg\sin\alpha}{\cos\alpha},$$

so $\omega = \sqrt{3g/(l\cos\alpha)}$ and $\alpha = \arccos(3g/(l\omega^2))$.

Solution to Exercise 15

(a) Use a polar coordinate system in the horizontal plane of the motion of particle A, with origin O at the centre of the circular motion, and take $\mathbf{k}$ to be a unit vector pointing vertically upwards. The forces on particle A are its weight $\mathbf{W}_1$ and the tension force $\mathbf{T}_1$ due to the string. The forces on particle B are its weight $\mathbf{W}_2$ and the tension force $\mathbf{T}_2$ due to the string. The situation is shown in the figure below.



Applying Newton's second law and equation (23) to particle A gives

$$-mR\omega^2\mathbf{e}_r = \mathbf{W}_1 + \mathbf{T}_1,$$

where $\mathbf{T}_1 = -(|\mathbf{T}_1|\sin\alpha)\mathbf{e}_r + (|\mathbf{T}_1|\cos\alpha)\mathbf{k}$ and $\mathbf{W}_1 = -mg\mathbf{k}$.

Since particle B is in equilibrium, we have

$$\mathbf{W}_2 + \mathbf{T}_2 = \mathbf{0},$$

where $\mathbf{W}_2 = -2mg\mathbf{k}$ and $\mathbf{T}_2 = |\mathbf{T}_2|\mathbf{k}$.

Resolving the above equations in the **k**-direction gives, respectively,

$$0 = -mg + |\mathbf{T}_1| \cos\alpha,$$
$$-2mg + |\mathbf{T}_2| = 0,$$

so $|\mathbf{T}_1| = mg/\cos\alpha$ and $|\mathbf{T}_2| = 2mg$.

But $|\mathbf{T}_1| = |\mathbf{T}_2|$, so $mg/\cos\alpha = 2mg$, giving $\cos\alpha = \frac{1}{2}$ and $\alpha = \frac{\pi}{3}$.

(b) Resolving the particle A equation in the $\mathbf{e}_r$-direction gives

$$-mR\omega^2 = -|\mathbf{T}_1| \sin\alpha,$$

so, using $|\mathbf{T}_1| = mg/\cos\alpha$ and $\alpha = \frac{\pi}{3}$, we have

$$\omega = \sqrt{\sqrt{3}g/R}.$$

Solution to Exercise 16

(a) Using equation (31) with $\mu = 1.3$, the maximum speed with which the car can follow a circular path of radius $R = 10$ is

$$v = \sqrt{\mu R g} = \sqrt{1.3 \times 10 \times 9.81} \simeq 11.29.$$

So the maximum speed here is about $11.29\,\mathrm{m\,s^{-1}}$.

(b) Rearranging equation (31) with $v = 15$, the condition to avoid slipping is

$$R \geq \frac{v^2}{\mu g} = \frac{15^2}{1.3 \times 9.81} \simeq 17.64.$$

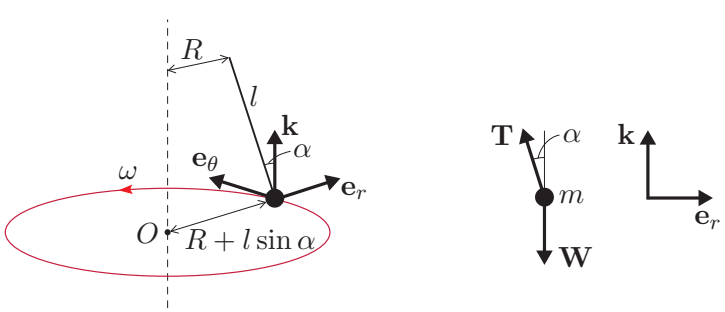
So the radius of the roundabout needs to be at least 18 m.

(c) We use the same condition as in part (b), but with $\mu = \frac{1}{2} \times 1.3 = 0.65$. So we need $R \geq 35.29$.

In this case, the roundabout needs a minimum radius of about 36 m.

Solution to Exercise 17

Use a polar coordinate system in the horizontal plane of the particle's motion, with origin at the centre of the circular motion, and take **k** to be a unit vector pointing vertically upwards. The radius of the uniform circular motion of the particle is $R + l\sin\alpha$. The forces on the particle are its weight **W** and the tension force **T** due to the string. The situation is shown in the figure below.



Applying Newton's second law and equation (23) to the particle gives

$$-m(R + l\sin\alpha)\omega^2\mathbf{e}_r = \mathbf{W} + \mathbf{T},$$

where $\mathbf{T} = -|\mathbf{T}|\sin\alpha\,\mathbf{e}_r + |\mathbf{T}|\cos\alpha\,\mathbf{k}$ and $\mathbf{W} = -mg\mathbf{k}$.

Resolving in the $\mathbf{e}_r$- and $\mathbf{k}$-directions gives, respectively,

$$-m(R + l\sin\alpha)\omega^2 = -|\mathbf{T}|\sin\alpha,$$
$$0 = -mg + |\mathbf{T}|\cos\alpha,$$

so $|\mathbf{T}| = mg/\cos\alpha$.

Substituting for $|\mathbf{T}|$ gives the equation of motion

$$m(R + l\sin\alpha)\omega^2 = mg\tan\alpha.$$

Hence, dividing through by $m\omega^2$, we have

$$\frac{g}{\omega^2}\tan\alpha = R + l\sin\alpha.$$

Solution to Exercise 18

This is obtained by substituting $\dot\theta = 0$ and $\theta = \pi$ into equation (41), which yields

$$\frac{v_0^2}{l^2} - \frac{4g}{l} = 0,$$

or

$$v_0^2 = 4gl,$$

so

$$v_0 = 2\sqrt{gl}.$$

Alternatively, conservation of energy can be used so that the result is obtained by equating loss of kinetic energy, $\frac{1}{2}mv_0^2$, with gain in potential energy, $2mgl$, so that $\frac{1}{2}v_0^2 = 2gl$ and hence the result.

Solution to Exercise 19

The essential difference between a model rod and model string is that a string needs to be kept under tension (i.e. with $T > 0$) in order for it to stay taut. For low speeds (at the lowest point), both pendulums will oscillate gently back and forth about the stable equilibrium position. For very high speeds, both will move in a circle around the fixed point. For intermediate speeds, the original pendulum will perform large oscillations or will circle around the fixed point, but the new pendulum will behave in this way only while the string stays taut (when the bob is above the level of the fixed point). If the speed is not high enough to keep the string taut, then the only force acting on the bob will be its weight, and the bob will behave like a projectile until the string becomes taut once more.

Solution to Exercise 20

(a) From equation (44), the string will not go slack provided that

$$T = \frac{mv_0^2}{l} + mg(3\cos\theta - 2) > 0.$$

Dividing by m/l and rearranging, we obtain

$$v_0^2 > gl(2 - 3\cos\theta).$$

The greatest value of $gl(2 - 3\cos\theta)$ occurs when $\cos\theta = -1$ (corresponding to $\theta = \pm\pi$), giving $v_0^2 > 5gl$ and $v_0 > \sqrt{5gl}$. So the string never goes slack if the speed v_0 of the pendulum bob when $\theta = 0$ is greater than $\sqrt{5gl}$ (in which case the bob performs complete revolutions).

(b) If $v_0 = 2\sqrt{gl}$, then from equation (41) we have

$$\begin{aligned}\dot{\theta}^2 &= \frac{4g}{l} - \frac{2g}{l}(1 - \cos\theta)\\ &= \frac{2g}{l}(1 + \cos\theta),\end{aligned}$$

so $\dot{\theta} = 0$ only when $\cos\theta = -1$, that is, when $\theta = \pi$. So the bob will reach the top of the circle unless the string goes slack first. We know from part (a) that for the string never to go slack, we must have $v_0 > \sqrt{5gl}$. Therefore since $2\sqrt{gl} < \sqrt{5gl}$, we know that the string must go slack before the bob comes momentarily to rest.

To see when the string goes slack, substitute $v_0 = 2\sqrt{gl}$ into equation (44) to obtain

$$\begin{aligned}T &= 4mg + mg(3\cos\theta - 2)\\ &= mg(3\cos\theta + 2)\\ &= 0.\end{aligned}$$

So the string goes slack when $3\cos\theta + 2 = 0$, which gives $\cos\theta = -2/3$. In the range $-\pi \leq \theta \leq \pi$, this gives θ as approximately ± 2.3 radians or $\pm 132°$. So the string goes slack when the pendulum makes an angle of approximately $42°$ above the horizontal.

Solution to Exercise 21

The condition for the pendulum bob to just reach the horizontal is $\dot{\theta} = 0$ at $\theta = \pm\frac{\pi}{2}$. So from equation (41), we find that

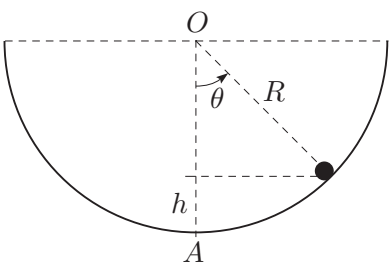
$$0 = \frac{v_0^2}{l^2} - \frac{2g}{l}.$$

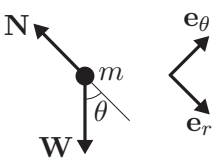
Hence $v_0 = \sqrt{2gl}$.

Alternatively, this result could be obtained by equating loss of kinetic energy, $\frac{1}{2}mv_0^2$, with gain in potential energy, mgl, so that $\frac{1}{2}v_0^2 = gl$ and hence the result.

Solution to Exercise 22

(a) Use a polar coordinate system in the plane of motion of the particle, with origin O at the centre of the circular motion. The forces acting on the particle are its weight $\mathbf{W}$ and the normal reaction $\mathbf{N}$ of the bowl. Let θ be the angle between the radius to the particle and the downward vertical, measured anticlockwise from the vertical, and let h be the vertical height of the particle above A. The situation is shown in the figure below.





The analysis of this situation is identical to that for the pendulum at the beginning of Section 3, with $\mathbf{T}$ replaced by $\mathbf{N}$ and l by R.

From equation (41), we have

$$v^2 = v_0^2 - 2gR(1 - \cos\theta),$$

where v_0 is the speed of the particle at A, and $v = R|\dot{\theta}|$ is its speed at angle θ. Initially, we have $h = \frac{1}{2}R$, $v = \sqrt{gR}$ and $\cos\theta = (R - h)/R = \frac{1}{2}$, giving

$$gR = v_0^2 - 2gR \times \tfrac{1}{2},$$

so $v_0^2 = 2gR$. Hence the first equation becomes

$$v^2 = 2gR\cos\theta.$$

The speed at the top of the bowl, where $\theta = \pm\frac{\pi}{2}$, is thus zero, showing that the particle just reaches this level.

(b) From equation (36), we have

$$|\mathbf{N}| = mg\cos\theta + mR\dot{\theta}^2.$$

From the equation at the end of the solution to part (a), we have

$$\dot{\theta}^2 = v^2/R^2 = (2g\cos\theta)/R,$$

so

$$|\mathbf{N}| = mg\cos\theta + 2mg\cos\theta = 3mg\cos\theta.$$

(We could, alternatively, have obtained this expression for $|\mathbf{N}|$ by use of equation (44) with $v_0^2 = 2gR$ and $l = R$.)

With $h = \frac{2}{3}R$, we have $\cos\theta = (R - h)/R = \frac{1}{3}$, so $|\mathbf{N}| = mg$, that is, the magnitude of the normal reaction when $h = \frac{2}{3}R$ is mg.

Solution to Exercise 23

Use a polar coordinate system in the vertical plane of the particle's motion, with origin O at the centre of the sphere. The forces acting on the particle are its weight $\mathbf{W}$ and the normal reaction $\mathbf{N}$ of the sphere. The situation is shown in the figure below.

The analysis of this situation is similar to that for the pendulum, with $\mathbf{T}$ replaced by $-\mathbf{N}$, $\mathbf{W}$ replaced by $-\mathbf{W}$, and l replaced by R, so we can make use of the results obtained for the pendulum at the beginning of the section.

With respect to the chosen coordinate system, we have

$$\mathbf{W} = -(mg\cos\theta)\mathbf{e}_r + (mg\sin\theta)\mathbf{e}_\theta,$$
$$\mathbf{N} = |\mathbf{N}|\mathbf{e}_r.$$

The acceleration of the particle is

$$\ddot{\mathbf{r}} = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta.$$

Applying Newton's second law to the particle gives

$$m\ddot{\mathbf{r}} = \mathbf{W} + \mathbf{N},$$

so we have

$$-mR\dot{\theta}^2\mathbf{e}_r + mR\ddot{\theta}\mathbf{e}_\theta = -(mg\cos\theta)\mathbf{e}_r + (mg\sin\theta)\mathbf{e}_\theta + |\mathbf{N}|\mathbf{e}_r.$$

Resolving in the $\mathbf{e}_r$-direction gives

$$-mR\dot{\theta}^2 = |\mathbf{N}| - mg\cos\theta,$$

and rearranging this equation yields

$$|\mathbf{N}| = mg\cos\theta - mR\dot{\theta}^2.$$

We could use the appropriately amended version of equation (41), with $l = R$, to substitute for $\dot{\theta}^2$ in this equation. Alternatively, we can use conservation of energy to obtain an expression for $\dot{\theta}^2$ as follows. The potential energy is $U = mgR(\cos\theta - 1)$, taking the highest point of the sphere ($\theta = 0$) as the datum. So the total conserved mechanical energy is

$$E = \tfrac{1}{2}mv^2 + mgR(\cos\theta - 1),$$

where $v = R|\dot{\theta}|$.

The value of E can be determined from the initial condition where $v = v_0$ at $\theta = 0$. Hence we have

$$\tfrac{1}{2}mR^2\dot{\theta}^2 + mgR(\cos\theta - 1) = \tfrac{1}{2}mv_0^2,$$

or

$$mR\dot{\theta}^2 = \frac{mv_0^2}{R} + 2mg(1 - \cos\theta),$$

and substituting into the equation for $|\mathbf{N}|$ gives

$$\begin{aligned}|\mathbf{N}| &= mg\cos\theta - \frac{mv_0^2}{R} - 2mg(1 - \cos\theta)\\ &= mg(3\cos\theta - 2) - \frac{mv_0^2}{R}.\end{aligned}$$

For $v_0 = \sqrt{\tfrac{1}{2}gR}$, we have

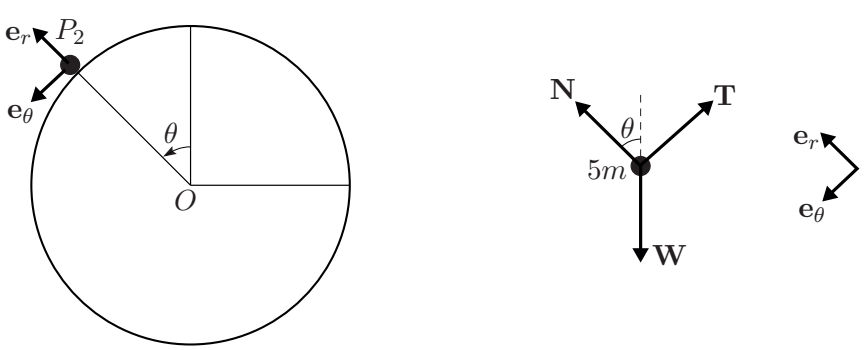
$$|\mathbf{N}| = mg(3\cos\theta - 2) - \tfrac{1}{2}mg = mg\big(3\cos\theta - \tfrac{5}{2}\big).$$

The particle will leave the surface when $\mathbf{N} = \mathbf{0}$, that is, when $3\cos\theta - \frac{5}{2} = 0$, which gives

$$\theta = \arccos\tfrac{5}{6} \simeq 0.59 \ (\simeq 33.6^\circ).$$

Solution to Exercise 24

(a) Use a polar coordinate system with θ being the angle that OP_2 makes with the upward vertical, similar to the set-up in Exercise 23, with the polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ defined with respect to P_2 as indicated in the figure below.



The forces acting on particle P_2 are its weight $\mathbf{W}$, the normal reaction of the cylinder $\mathbf{N}$, and the tension of the string $\mathbf{T}$. These forces are indicated in the figure and are expressed in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as

$$\begin{aligned}\mathbf{N} &= |\mathbf{N}|\mathbf{e}_r,\\ \mathbf{T} &= -|\mathbf{T}|\mathbf{e}_\theta,\\ \mathbf{W} &= 5mg(\mathbf{e}_\theta\sin\theta - \mathbf{e}_r\cos\theta).\end{aligned}$$

The acceleration of P_2 is

$$\ddot{\mathbf{r}} = R\ddot{\theta}\mathbf{e}_\theta - R\dot{\theta}^2\mathbf{e}_r$$

(as in Exercise 23). Newton's second law applied at P_2 gives

$$5m\ddot{\mathbf{r}} = \mathbf{W} + \mathbf{N} + \mathbf{T},$$

so we have

$$5mR(\ddot{\theta}\mathbf{e}_\theta - \dot{\theta}^2\mathbf{e}_r) = 5mg(\mathbf{e}_\theta \sin\theta - \mathbf{e}_r\cos\theta) + |\mathbf{N}|\mathbf{e}_r - |\mathbf{T}|\mathbf{e}_\theta,$$

and resolving in the $\mathbf{e}_r$-direction yields

$$-5mR\dot{\theta}^2 = |\mathbf{N}| - 5mg\cos\theta,$$

which can be rearranged to give the required result.

(b) The total mechanical energy of the combined system is conserved since the gravitational forces are conservative forces and the sum of the contributions of the string tensions to work done on the particles vanishes as a result of Newton's third law.

The total mechanical energy E is the sum of the energies of both particles, P_1 and P_2. The kinetic energy of P_1 is $\frac{1}{2} \times 3m \times v^2$, and taking the horizontal diameter through O as the datum, its potential energy is $3mgR\sin\theta$. Similarly, the kinetic energy of P_2 is $\frac{1}{2} \times 5m \times v^2$, and its potential energy is $5mgR\cos\theta$. Thus the total mechanical energy of both particles is

$$\begin{aligned} E &= \tfrac{3}{2}mv^2 + 3mgR\sin\theta + \tfrac{5}{2}mv^2 + 5mgR\cos\theta \\ &= 4mv^2 + mgR(3\sin\theta + 5\cos\theta), \end{aligned}$$

which stays constant throughout the motion (provided that both particles stay on the cylinder).

The value of E can be determined by the initial condition where $v = 0$ when $\theta = \pi/4$. Given that $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$, we have

$$E = mgR(8/\sqrt{2}) = 4\sqrt{2}mgR,$$

and substituting this into the equation above and dividing through by m gives

$$4v^2 + gR(3\sin\theta + 5\cos\theta) = 4\sqrt{2}gR,$$

which can be rearranged to give the required result.

(c) Recalling that $v = R|\dot{\theta}|$, the result of part (b) gives

$$R\dot{\theta}^2 = \frac{v^2}{R} = \tfrac{1}{4}g(4\sqrt{2} - 5\cos\theta - 3\sin\theta),$$

which can be substituted into the expression for $|\mathbf{N}|$ obtained in part (a) to give

$$\begin{aligned} |\mathbf{N}| &= 5m\left(g\cos\theta - \tfrac{1}{4}g(4\sqrt{2} - 5\cos\theta - 3\sin\theta)\right) \\ &= \tfrac{5}{4}mg(9\cos\theta + 3\sin\theta - 4\sqrt{2}). \end{aligned}$$

Particle P_2 will leave the cylinder when $|\mathbf{N}| = 0$, that is, when

$$9\cos\theta + 3\sin\theta = 4\sqrt{2},$$

as required. The left-hand side of this equation can be expressed as

$$9\cos\theta + 3\sin\theta = A\cos(\theta - \phi)$$

with

$$A\cos\phi = 9 \quad \text{and} \quad A\sin\phi = 3,$$

so $A^2 = 9^2 + 3^2 = 90$ or $A = 3\sqrt{10}$, and $\tan\phi = 3/9 = 1/3$ or $\phi = \arctan(1/3)$. Thus we have

$$3\sqrt{10}\cos(\theta - \phi) = 4\sqrt{2},$$

or

$$\theta - \phi = \arccos\left(\tfrac{4}{3\sqrt{5}}\right),$$

and with $\phi = \arctan(1/3)$ we obtain the result $\theta \simeq 1.254$ (in radians), which is equivalent to 71.8° to three significant figures.

Solution to Exercise 25

(a) Since the motion is clockwise, θ is decreasing during the motion, so $\dot{\theta} = -2$ and the particle's angular velocity is

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{k} = -2\mathbf{k}.$$

(b) From equation (46), the particle's velocity is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = -2\mathbf{k} \times (3\mathbf{i} + 4\mathbf{j}) = 8\mathbf{i} - 6\mathbf{j}.$$

Since the radius of the circular path is $\sqrt{3^2 + 4^2} = 5$, the velocity is also given by

$$\mathbf{v} = \dot{\mathbf{r}} = R\dot{\theta}\mathbf{e}_\theta = 5 \times (-2)\mathbf{e}_\theta = -10\mathbf{e}_\theta.$$

Alternatively, we have

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = -2\mathbf{k} \times 5\mathbf{e}_r = -10\mathbf{e}_\theta.$$

Solution to Exercise 26

By definition, the angular momentum is

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}} = (\mathbf{i} + 2\mathbf{j}) \times 3(2\mathbf{i} - \mathbf{j}) = (-3\mathbf{i} \times \mathbf{j}) + (12\mathbf{j} \times \mathbf{i}) = -15\mathbf{k},$$

where we have used $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{j} \times \mathbf{i} = -\mathbf{i} \times \mathbf{j} = -\mathbf{k}$.

Solution to Exercise 27

We know, from the text just before the exercise, that $\mathbf{L} = mR^2\dot{\theta}\mathbf{k}$, so $|\mathbf{L}| = mR^2|\dot{\theta}|$. We also know, from Section 1, that $\mathbf{r} = R\mathbf{e}_r$ and $\dot{\mathbf{r}} = R\dot{\theta}\mathbf{e}_\theta$, so $|\mathbf{r}| = R$ and $|\dot{\mathbf{r}}| = R|\dot{\theta}|$. Hence we have

$$m|\mathbf{r}|\,|\dot{\mathbf{r}}| = mR^2|\dot{\theta}| = |\mathbf{L}|.$$

Solution to Exercise 28

The position of the particle is

$$\mathbf{r} = a\cos(\alpha t)\mathbf{i} + a\sin(\alpha t)\mathbf{j} + ut\mathbf{k}.$$

Hence the linear momentum of the particle is

$$\mathbf{p} = m\dot{\mathbf{r}} = m(-a\alpha\sin(\alpha t)\mathbf{i} + a\alpha\cos(\alpha t)\mathbf{j} + u\mathbf{k}).$$

Therefore the force acting on the particle is

$$\mathbf{F} = \dot{\mathbf{p}} = -ma\alpha^2(\cos(\alpha t)\mathbf{i} + \sin(\alpha t)\mathbf{j}).$$

The angular momentum of the particle about the origin O is

$$\begin{aligned}\mathbf{L} &= \mathbf{r} \times m\dot{\mathbf{r}}\\ &= (a\cos(\alpha t)\mathbf{i} + a\sin(\alpha t)\mathbf{j} + ut\mathbf{k})\\ &\quad \times m(-a\alpha\sin(\alpha t)\mathbf{i} + a\alpha\cos(\alpha t)\mathbf{j} + u\mathbf{k})\\ &= mau(\sin(\alpha t) - \alpha t\cos(\alpha t))\mathbf{i} - mau(\alpha t\sin(\alpha t) + \cos(\alpha t))\mathbf{j}\\ &\quad + ma^2\alpha(\cos^2(\alpha t) + \sin^2(\alpha t))\mathbf{k}\\ &= mau(\sin(\alpha t) - \alpha t\cos(\alpha t))\mathbf{i} - mau(\cos(\alpha t) + \alpha t\sin(\alpha t))\mathbf{j}\\ &\quad + ma^2\alpha\,\mathbf{k}.\end{aligned}$$

The torque about O acting on the particle is

$$\begin{aligned}\boldsymbol{\Gamma} &= \mathbf{r} \times \mathbf{F}\\ &= -(a\cos(\alpha t)\mathbf{i} + a\sin(\alpha t)\mathbf{j} + ut\mathbf{k}) \times ma\alpha^2(\cos(\alpha t)\mathbf{i} + \sin(\alpha t)\mathbf{j})\\ &= ma\alpha^2(ut\sin(\alpha t)\mathbf{i} - ut\cos(\alpha t)\mathbf{j})\\ &= mau\alpha^2 t(\sin(\alpha t)\mathbf{i} - \cos(\alpha t)\mathbf{j}).\end{aligned}$$

To verify the torque law, we need to check that $\dot{\mathbf{L}} = \boldsymbol{\Gamma}$. We have

$$\begin{aligned}\dot{\mathbf{L}} &= mau(\alpha\cos(\alpha t) - (\alpha\cos(\alpha t) - \alpha^2 t\sin(\alpha t)))\mathbf{i}\\ &\quad - mau(-\alpha\sin(\alpha t) + (\alpha\sin(\alpha t) + \alpha^2 t\cos(\alpha t)))\mathbf{j}\\ &= mau\alpha^2 t(\sin(\alpha t)\mathbf{i} - \cos(\alpha t)\mathbf{j})\\ &= \boldsymbol{\Gamma},\end{aligned}$$

as required.

Solution to Exercise 29

(a) Let $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$ be the angular velocity of the particle. From Example 5 and the discussion that follows it, the angular momentum is $\mathbf{L} = mR^2\dot{\theta}\mathbf{k}$. The torque law gives $\dot{\mathbf{L}} = \boldsymbol{\Gamma}$, so we have

$$mR^2\ddot{\theta}\mathbf{k} = \Gamma_0 e^{-\alpha t}\mathbf{k}.$$

Resolving in the $\mathbf{k}$-direction and rearranging, we obtain

$$\ddot{\theta} = \frac{\Gamma_0}{mR^2}\,e^{-\alpha t}.$$

Integrating, we obtain

$$\dot{\theta} = -\frac{\Gamma_0}{\alpha mR^2}\,e^{-\alpha t} + C.$$

Taking $t = 0$ when the house is first picked up by the whirlwind, so $\dot{\theta}(0) = 0$, we have $C = \Gamma_0/(\alpha m R^2)$ and hence

$$\dot{\theta} = \frac{\Gamma_0}{\alpha m R^2}\,(1 - e^{-\alpha t}).$$

So the angular speed is

$$\omega = |\dot{\theta}| = \frac{\Gamma_0}{\alpha m R^2}\,(1 - e^{-\alpha t}).$$

As $t \to \infty$, $\omega \to \Gamma_0/(\alpha m R^2)$.

(b) Assuming that the model remains valid for decreasing R, as R becomes smaller, ω becomes larger and larger without limit. However, as R becomes smaller, the distance from the centre of mass of the house to the axis of the whirlwind gets smaller and smaller. Eventually, some parts of the house will coincide with this axis and the particle model will no longer be valid because the dimensions of the house can no longer be neglected.

Solution to Exercise 30

The total torque about the origin is given by

$$\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times f(r)\,\widehat{\mathbf{r}} = \mathbf{0},$$

since $\mathbf{r}$ and $\widehat{\mathbf{r}}$ are parallel. So the total torque about the origin is zero.

Solution to Exercise 31

Choose the origin to be the centre of the circle. Let ω be the (constant) angular speed of the particle, and let m be its mass. Let R be the radius of the circular motion. Using equation (23), the total force acting on the particle is $-mR\omega^2\mathbf{e}_r$, so it is a central force. Hence, from the result of Exercise 30, the total torque about the centre of the circle is zero.

Acknowledgements

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